The Decision Problem and the Development of Metalogic

Richard Zach

University of Calgary, Canada
Department of Philosophy and
Calgary Institute for the Humanities
www.ucalgary.ca/~rzach/

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Outline

Introduction

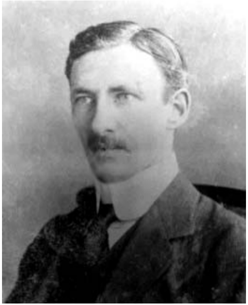
- Decision Problem: find a procedure which decides after finitely many steps if a given formula is valid or not
- One of the fundamental problems for the development of modern logic in the 1920s and 1930s
- Intimately tied up with
  - Hilbert's philosophical project
  - development of model theory
  - development of notion of decision procedure/computation

David Hilbert, 1862–1943

- Studied mathematics at the University of Königsberg
- Became chair of the Mathematics Department at Göttingen in 1895
- First lectures on logic in 1905
- Program in foundations from 1917 onward[a]
- *Principles of Mathematical Logic*, 1917/1918

[a]Sieg, Hilbert's programs, *BSL* 1999

Paul Bernays, 1888–1977



- Dissertation in 1912 on analytic number theory in Göttingen
- Assistant to Hilbert from 1917 onward
- *Habilitation* in 1918 on the propositional calculus of *Principia*; proved completeness[a]
- Had to leave Germany in 1933; moved to Zurich
- Hilbert-Bernays, *Foundations of Mathematics* (1934, 1939)

[a] Zach, Completeness before Post, *BSL* 1999

Heinrich Behmann, 1891–1970

- Studied mathematics under Hilbert
- Dissertation in 1918 on *Principia Mathematica*[a]
- *Habilitation* in 1921 on the decision problem
- Lectured on logic in Göttingen 1923
- Moved to Halle-Wittenberg in 1925
- Dismissed in 1945

[a] Mancosu, Between Russell and Hilbert, *BSL* 1999

Moses Schönfinkel, 1889–1942

- Worked in Göttingen 1914–1924
- Invented combinatory logic as an extension of *Principia* around 1920 (edited by Behmann, published 1924)
- Work on decision problem in 1922, expanded and published by Bernays in 1928

Wilhelm Ackermann, 1896–1962

- Dissertation in 1924 on Hilbert's ε-substitution method[a]
- Discovered the Ackermann function in 1925
- Hilbert-Ackermann, *Grundzüge der theoretischen Logik*, 1928
- Proved decidability of the Ackermann class in 1928
- Taught in *Gymnasium* from 1929 onward

[a] Zach, The practice of finitism, *Synthèse* 2003

Early Work

- Attempt at a decidability proof for propositional logic in Hilbert's 1905 lectures on logic.
- Hilbert's 1917/18 lecture course "Principles of Mathematics" contain a completeness proof for propositional logic. Uses normal forms in essential way.
- From this completeness result, decidability of the propositional calculus follows; a conclusion first explicitly drawn in Bernays's 1918 *Habilitationsschrift*.

The Decision Problem

Behmann, "Entscheidungsproblem und Algebra der Logik"
Göttingen, May 10, 1921

- Aim in mathematics to find *general procedures* for *wide classes* of problems which do not require inventive ingenuity but only mastery of calculatory tools. Transform problems to mere *calculatory exercise*.
- For *proof of mathematical assertions*, this aim is far from achieved: in proving theorems, new approaches required for each case.
- Path to solution: axiomatization of mathematics using logical formalism. This turns mathematical proof into a calculatory exercise, into a game. But: *the rules of the game only tell us what one may do and not what one should do.*

The Decision Problem

"We require a lot more: that not only the individual operations but also the *path of calculation* as a whole should be specified by rules, in other words, *an elimination of thinking in favor of mechanical calculation.*

If a logical or mathematical assertion is given, the required procedure should give complete instructions for determining whether the assertion is *correct or false by a determinate calculation after finitely many steps*.

The problem thus formulated I want to call the *general decision problem*."

Methods of Proof

"For the character of the [decision] problem it is fundamental that the only tool that is allowed for the proof is *completely mechanical calculation* following a given instruction, without any activity of thinking in the stricter sense.

One might, if one wanted to, speak of *mechanical* or *machine-like thinking*.

(Perhaps one can later on even let it be carried out by a machine.)"

Methods of Proof

- "Calculation *means* deterministic manipulation of symbols" (lectures from 1922/23)
- Connection to algebra, including algebra of logic; symbolism of Schröder's algebra of logic thought inadequate

Formula Classes

Aim: show every 2nd order monadic formula in can be transformed into a boolean combination of conditions on the size of the domain: "there are at least n objects" or "there are at most n objects"

Normal Forms in A

- Every Boolean combination of formulas can be brought into one of two normal forms by distributing:
  - Disjunctive normal form: $\bigvee \bigwedge A_i$
  - Conjunctive normal form: $\bigwedge \bigvee A_i$
- Every formula can be brought into prenex form

  $$(\forall x) \ldots (\exists y) \ldots (\forall z) A$$

  where A is quantifier-free.
- Quantifiers can also be pushed in: all $(\forall x)$ are in front of disjunctions of literals ($P(x)$ or $\neg P(x)$) and all $\exists$ are in front of conjunctions of literals.
- (Decision procedure for quantified Boolean formulas: $(\forall p)p$ is false, $(\forall p)\neg p$ is true, …)

Elimination of 2nd Order Quantifiers

- Basic idea: consider innermost subformula $(\exists P)A(P)$ where $A(P)$ only contains first order quantifiers.
- Push $(\exists P)$ suitably into the formula so that
- innermost subformulas are $(\exists P)A'(P)$ with $A'(P)$ in special form.
- Observe that $(\exists P)A'(P)$ equivalent to first-order formula

Elimination of 2nd Order Quantifiers

- Consider

$$(\exists P)[(\forall x)(\neg A(x) \vee P(x)) \wedge (\forall x)(B(x) \vee \neg P(x))]$$

- This is equivalent to

$$(\forall x)(\neg A(x) \vee B(x))$$

- Why? Rewrite using $\to$:

$$(\exists P)[(\forall x)(A(x) \to P(x)) \wedge (\forall x)(P(x) \to B(x))]$$

Elimination of 2nd Order Quantifiers

- Consider

$$(\exists P)[(\exists x)(A(x) \wedge P(x)) \wedge (\exists y)(B(y) \wedge \neg P(y))]$$

- This is equivalent to

$$(\exists x)(\exists y)(\exists P)[(A(x) \wedge P(x)) \wedge (B(y) \wedge \neg P(y))]$$

- In turn equivalent to

$$(\exists x)(\exists y)[A(x) \wedge B(y) \wedge x \neq y]$$

- Reduce such claims to claims about number of elements of domain

Hilbert and Bernays on Decision Problem

Lectures "Logische Grundlagen der Mathematik", co-taught by Hilbert and Bernays, Winter 1922/23

- Statement of decision problem
- Discussion of importance, examples from geometry
- New proof of decidability of first-order monadic logic (without =)
- Parts of relevant section of lecture notes verbatim in Hilbert/Ackermann 1928, §11

Finite Controllability

- To show that a monadic formula is valid, it suffices to show that if it were not valid, i.e., is false in some interpretation, it is false in a finite interpretation.
- Suppose A is false in an interpretation. If A contains k predicate symbols $P_1, \ldots, P_n$, then divide the elements of the domain into classes: a and b belong to the same class whenever $P_i(a)$ iff $P_i(b)$ for $i = 1, \ldots, k$. There can be at most 2^k classes.

Finite Controllability

- Define a new interpretation where the domain consists of these classes. If A is false in the old interpretation, it is also false in the new interpretation (with at most 2^k elements).
- To decide if a formula is valid, test if it is valid in all interpretations on 2^k elements.

The Decision Problem and Consistency

- For specific theories (e.g., geometry), questions of provability can be solved using decision procedure (were it available), e.g.,
  - B not provable from axioms A of geometry if $A \to B$ not valid.
  - Consistency: A is consistent if $\neg A$ not valid.
- Converses require completeness theorem; unclear if H/B realized this

The Decision Problem and Completeness

- Completeness in syntactic sense:

  A is complete if, whenever unprovable formula added to the axioms, system becomes inconsistent.
- Problem of completeness "is neither solved by the decision problem, nor does it contain it"

Schönfinkel's Original Proof

- Manuscript (in Bernays Nachlass) from Winter term 1922/23
- Treats case of validity of first-order formulas of the form $(\exists x)(\forall y)A$ where A only contains one binary predicate symbol
- Uses infinite sums and products!

Bernays and Schönfinkel 1927

"Zum Entscheidungsproblem der mathematischen Logik", *Math. Ann.* 99 (1928)

- Discussion of decision problem in general, cardinality questions in particular
- Explicit discussion of relation between satisfiability and validity
- Discussion of satisfiability and finite satisfiability (Löwenheim and Skolem)
- Proof of finite controllability of monadic class with bound (from 1922/23 lectures)
- Discussion of prefix classes

The Bernays-Schönfinkel Class

- The class $\forall^*\exists^* A$ is shown to be decidable for validity in Bernays and Schönfinkel 1928
- It is the preliminary "trivial" case before Bernays goes on to the actual contribution of the paper
- The Bernays-Schönfinkel class is not dealt with in Schönfinkel's manuscript (although special cases $\forall\forall$, $\exists\exists$, and $\forall\exists$ are mentioned and claimed to be trivial).

The Bernays-Schönfinkel Class

- Consider a formula $(\forall x)(\forall y)(\forall z)(\exists u)(\exists v)A(x, y, z, u, v)$.
- If this is valid in domains with at least 3 elements, the disjunction

$$A(a, b, c, a, a) \lor A(a, b, c, a, b) \lor A(a, b, c, a, c) \lor \ldots$$
$$\ldots \lor A(a, b, c, c, c)$$

  must be a tautology.
- If it is a tautology, the original formula is valid (in all domains).
- Special case of Herbrand's Theorem: reduction to propositional calculus.

The Schönfinkel Class

- $(\exists x)(\forall y)A$ with A quantifier-free and containing n binary predicate symbols decidable for validity
- proof is by giving bound on countermodel: 2^n

The Ackermann Class

"Über die Erfüllbarkeit gewisser Zählausdrücke", *Math. Ann.* 100 (1929)

- Deals with formulas of the form $\exists^*\forall\exists^*A$ and satisfiability
  (i.e., $\forall^*\exists\forall^*A$ and validity)
- First paper that emphasizes satisfiability and upper bound on model

The Development of Semantics

- Truth-functional semantics for the propositional calculus already in H 1917/18 and B 1918 (explicit with completeness proof).
- Semantics for first-order logic in H 1917/18 vague.
- Two issues:
  - Domain of quantification
  - Interpretation of predicate variables
- "correct [richtig]" vs. "valid [allgemeingültig]"
- validity vs. satisfiability

Semantical Subtleties

- Behmann distinguishes predicate constants and variables; formulas with (free) predicate variables are not assertions
- Decide if formula is "richtig" for every substitution for variables (individual and predicate)
- Decide if formula is "richtig" for every domain of individuals
- Bernays/Schönfinkel only place where decision problem is used explicitly to cover both
- Also makes explicit that for "substitutions" only courses-of-values are to be considered.

Behmann: Monadic Second-order Logic

- Behmann 1921 and 1922
- decidability of *validity*
- proof still proceeds by giving determinate procedure for manipulating formulas
- reduction to propositional logic: elimination of quantifiers
- explicit connection with *mechanical calculability*, deterministic manipulation of symbols

Hilbert and Bernays: Monadic Predicate Logic

- Hilbert/Bernays 1922/23, Bernays/Schönfinkel 1928, Hilbert/Ackermann 1928:
- Giving bounds for size of countermodels
- In Bernays/Schönfinkel equivalence between decision problem for validity and satisfiability explicit

Bernays, Schönfinkel, Ackermann

- Emphasis on prenex formulas
- No syntactic procedures but bounds for size of interpretation
- Reduction to propositional logic for Bernays-Schönfinkel class
- Ackermann approach via satisfiability instead of validity

Influences

- Löwenheim 1915: known in Göttingen in 1922 (In review of Behmann 1922 by Bernays in *Jahrbuch für Fortschritte der Mathematik* 48.1119.02)
- Not discussed until 1928, but then explicitly by Bernays
- terminology "Zählausdruck" used by Ackermann
- Skolem 1920: not cited until Bernays/Schönfinkel 1928 and Hilbert/Ackermann, and only as "simplified proof of Löwenheim's theorem".

Future Work

- Trace connections with other work in and outside of Hilbert School:
  - Schröder—elimination problem in algebra of logic
  - Herbrand—connection to consistency
  - Gödel—completeness, decidability of Gödel class
  - Skolem—Skolem functions and normal form
- Settle relation between Löwenheim and Skolem and work in the Hilbert school in 1920s

Priorities and Credit

- Löwenheim did not know that he proved decidability of monadic class
- Behmann did not merely re-prove Löwenheim's result, but
  - gave more informative proof
  - proved decidability of monadic *second-order* logic
  - knew that he was proving a decidability result
- The Bernays-Schönfinkel class should really be called the "Bernays class."

For further reading

- Paolo Mancosu.
  Between Russell and Hilbert: Behmann on the foundations of mathematics.
  Bulletin of Symbolic Logic 5 (1999) 303–330.
- Paolo Mancosu, Richard Zach, and Calixto Badesa.
  The Development of Mathematical Logic from Russell to Tarski: 1900–1935.
  In: Leila Haaparanta, ed., *The History of Modern Logic*. OUP, to appear.
- Wilfried Sieg.
  Hilbert's Programs: 1917–1922. *Bulletin of Symbolic Logic* 5 (1999) 1–44.
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  The practice of finitism: Epsilon calculus and consistency proofs in Hilbert's Program. *Synthèse* 137 (2003) 211–259.

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