TOPOLOGICAL SEMANTICS FOR FIRST-ORDER MODAL LOGIC (ABSTRACT)

STEVE AWODEY* AND KOHEI KISHIDA**

1. Topological semantics for propositional S4

As McKinsey and Tarski [6] showed, the Stone representation theorem for Boolean algebras extends to algebras with operators, providing topological semantics for (classical) propositional modal logic. The "necessity" operator is interpreted as taking the open interior of an arbitrary subset of a topological space. Given a topological space X, choose an arbitrary subset (not necessarily open) $[\![P]\!]$ for each atomic sentence P, and set

$$\begin{aligned}
[\![\neg\varphi]\!] &= X - [\![\varphi]\!] \\
[\![\varphi \wedge \psi]\!] &= [\![\varphi]\!] \cap [\![\psi]\!] \\
[\![\varphi \vee \psi]\!] &= [\![\varphi]\!] \cup [\![\psi]\!] \\
[\![\top]\!] &= X \\
[\![\bot]\!] &= \varnothing \\
[\![\Box\varphi]\!] &= \mathrm{int}([\![\varphi]\!]).
\end{aligned}$$

Also define

$$(X, [\![\cdot]\!]) \vDash \varphi \iff [\![\varphi]\!] = X.$$

Propositional modal logic S4 is sound with respect to this semantics. It is indeed complete as follows.

Theorem 1. *For any consistent theory $\mathbb{T}$ of propositional modal logic containing S4, there is a topological interpretation $(X, [\![\cdot]\!])$ such that every φ has*

$$\mathbb{T} \vdash \varphi \iff (X, [\![\cdot]\!]) \vDash \varphi$$

We can prove this by taking X the set of ultrafilters of Lindenbaum algebras of $\mathbb{T}$, and topologizing X with basic open sets $[\![\Box\varphi]\!] = \{\, u \in X \mid [\varphi] \in u \,\}$ for all φ.

Date: May 11, 2006.

*Philosophy, Carnegie Mellon Univ.; awodey@cmu.edu.

**Philosophy, Univ. of Pittsburgh; kok6@pitt.edu.

2. Denotational formulation of semantics for first-order logic

Let us extend the notation $[\![\cdot]\!]$ from sentences to first-order formulas with variables: $[\![\, x_1, \ldots, x_n \mid \varphi \,]\!]$ means that variables $x_1, \ldots, x_n$ may occur freely in the formula φ but no other free variables. Then the usual semantics for first-order (non-modal) logic can be formulated with interpretation $[\![\cdot]\!]$, as follows: In a model of first-order logic $M = \langle |M|, R^M, f^M, c^M, \ldots, \rangle$ let $[\![\, x_1, \ldots, x_n \mid \varphi \,]\!]$ stand for the subset of $|M|^n$ defined by the formula φ, i.e.

$$[\![\, x_1, \ldots, x_n \mid \varphi \,]\!] = \{\, (a_1, \ldots, a_n) \in |M|^n \mid M \vDash \varphi(a_1, \ldots, a_n) \,\} \subseteq |M|^n.$$

Terms t can be interpreted in a similar manner, as definable functions $[\![t]\!] : |M|^n \to |M|$.

3. Topological semantics for first-order S4

The main result of this paper is to generalize the topological interpretation from propositional to first-order logic by replacing the space with a sheaf over it. Given a topological space X, a sheaf over X is a pair $D, \pi : D \to X$ such that D is a topological space and π is a local homeomorphism. That is, π is a continuous map such that every $a \in D$ has an open neighborhood $V \subseteq D$ with $\pi|_V : V \to \pi[V]$ a homeomorphism. Note that D is the disjoint union $\sum_{p \in X} D_p$ of all "fibers" $D_p = \pi^{-1}[\{p\}]$ for $p \in X$, each of which is a discrete subspace $D_p \subseteq D$.

Let us choose arbitrary subsets (not necessarily open) $[\![R]\!] \subseteq D^n$ for n-ary relation symbols R. Here D^n is the "fiberwise product" over X, i.e., $D^n = \sum_{p \in X} (\underbrace{D_p \times \cdots \times D_p}_{n\text{-times}})$. Also choose arbitrary maps of sheaves $[\![f]\!] : D^n \to D$ for n-ary function symbols f, as well as maps of sheaves $[\![c]\!] : X (= D^0) \to D$ for constants c. That is, $[\![f]\!]$ and $[\![c]\!]$ are *continuous* maps respecting fibers in the sense that $\pi([\![f]\!](a)) = \pi(a)$. Then, restricted to each fiber D_p,

$$\langle D_p, [\![R]\!]_p (\subseteq {D_p}^n), [\![f]\!]_p (: {D_p}^n \to D_p), [\![c]\!]_p (\in D_p), \ldots \rangle$$

is a model of first-order (non-modal) logic. We use the usual first-order semantics in each fiber and take a disjoint union to interpret the first-order part of first-order modal logic.

Then $\Box$ is interpreted by the interior operation in X and D. That is, $[\![\Box\sigma]\!] = \mathrm{int}[\![\sigma]\!] \subseteq X (= D^0)$ for sentences σ and $[\![\, \bar{x} \mid \Box\varphi \,]\!] = \mathrm{int}[\![\, \bar{x} \mid \varphi \,]\!] \subseteq D^n$ for formulas φ (with no more than n variables).

In order for this semantics to be well defined, we need to make sure that the interior operation commutes with substitution of terms, i.e.,

$$[\![t]\!]^{-1}[\mathrm{int}[\![\, x \mid \varphi \,]\!]] = [\![\, \bar{y} \mid \Box\varphi[t(\bar{y})/x] \,]\!] = \mathrm{int}([\![t]\!]^{-1}[[\![\, x \mid \varphi \,]\!]]),$$

since our syntax has $(\Box\varphi)[t/x] = \Box(\varphi[t/x])$. Also, well-defining $[\![\, x \mid \Box\varphi(x,x) \,]\!]$ from $[\![\, x, y \mid \varphi(x,y) \,]\!]$ requires Δ^{-1} commuting with int. These conditions require that (D, π) be a sheaf over X.

Let FoS4 be the straightforward union of FoL rules and propositional S4 rules, in which FoL-rule schemes do not discriminate modal formulas from non-modal ones. Its theorems include $\exists x \Box\varphi \vdash \Box\exists x \Box\varphi$, which means the projection π is an open map, and $x = y \vdash \Box x = y$, which means the diagonal Δ is an open map (these conditions make (D, π) a sheaf). Not only is FoS4 sound with respect to the topological semantics above (FoL rules and S4 rules can be separately checked for soundness), we have completeness in the strong form:

Theorem 2. *For any consistent theory $\mathbb{T}$ of first-order modal logic containing FoS4, there is a topological interpretation $(\pi : D \to X, [\![\cdot]\!])$ such that every φ has*

$$\mathbb{T} \vdash \varphi \iff (\pi, [\![\cdot]\!]) \vDash \varphi \ (\textit{i.e. } [\![\, \bar{x} \mid \varphi \,]\!] = D^n).$$

References

[1] S. Awodey and K. Kishida, "Topological Semantics of First-order Modal Logic," in preparation.

[2] M. Fairtlough, M. Mendler, and E. Moggi, eds., *FLoC Satellite Workshop on Intuitionistic Modal Logics and Applications (IMLA'99)*, Trento, Italy, July 1999.

[3] B. Hilken and D. Rydeheard, "A First Order Modal Logic and Its Sheaf Models," in [2], 1999.

[4] S. Mac Lane and I. Moerdijk, *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*, Springer-Verlag New York: New York, 1992.

[5] N.-Y. Suzuki, "Algebraic Kripke Sheaf Semantics for Non-Classical Predicate Logics," *Studia Logica* **63** (1999), 387–416.

[6] J. McKinsey and A. Tarski, "The Algebra of Topology," *Annals of Mathematics* **45** (1944), 141–91.