CONCEPT CALCULUS

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1. THE GENERAL NATURE OF CONCEPT CALCULUS.

Concept Calculus is a new mathematical/philosophical program of wide scope. The development of Concept Calculus began in Summer, 2006.

Concept Calculus promises to connect mathematics, philosophy, and commonsense thinking in a radically new way.

Advances in Concept Calculus are made through rigorous mathematical findings, and promise to be of immediate and growing interest to philosophers.

Developments in Concept Calculus generally consist of the following.

a. An identification of a few related concepts from commonsense thinking. In the various developments, the choice of these concepts will vary greatly. In fact, all concepts from ordinary language are prime targets.

b. Formulation of a variety of fundamental principles involving these concepts. These various principles may have various degrees of plausibility, and may even be incompatible with each other. There may be no agreement among philosophers as to just which principles to accept. Concept Calculus is concerned only with logical structure.

c. Formulation of a variety of systems of such fundamental principles in b. These systems generally combine several such fundamental principles in some attractive way.

d. An analysis of the "interpretation power" of these resulting systems.

e. In particular, for each of the resulting systems, a determination of whether they interpret mathematics - as formalized by ZFC.

A system T having interpretation power at least that of mathematics (ZFC) has special significance. This means that

if T is without contradiction then mathematics (ZFC) is without contradiction.

I.e., we have a consistency proof for mathematics (ZFC) relative to that of T.

Furthermore, relative consistency proofs arising this way are generally very finitary.

2. INTERPRETATION POWER.

The interest of Concept Calculus rests considerably on the significance of interpretation power.

Interpretability between formal systems was first precisely defined by Tarski.

We define this for systems S,T in first order predicate calculus with equality. S,T may have completely different symbols.

An interpretation of S in T consists of

i. A one place relation de-fined in T which is meant to carve out the domain of objects that S is referring to, from the point of view of T.

ii. A definition of the constants, relations, and functions in the language of S by formulas in the language of T, whose free variables are restricted to the domain of objects that S is refer-ring to (in the sense of i).

iii. It is required that every axiom of S, when translated into the language of T by means of i,ii, becomes a theorem of T.

In ii, we usually allow that the equality relation in S need not be interpreted as equality - but rather as an equivalence relation.

We give some examples.

S consists of the axioms for linear order, together with "there is a least element".

i. □(x < x).
ii. (x < y □ y < z) □ x < z.
iii. x < y   y < x   x = y.
iv. (□x)(□y)(x < y   x = y).

T consists of the axioms for linear order, together with "there is a greatest element".

i. □(x < x).
ii. (x < y □ y < z) □ x < z.
iii. x < y   y < x   x = y.
iv. (□x)(□y)(y < x   x = y).

Note that S,T are theories in first order predicate calculus with equality, in the same language: just the binary relation symbol <.

CLAIM: S is interpretable in T and T is interpretable in S. They are mutually interpretable.

Here is the obvious interpretation of S in T. In T, take the objects of S to be everything (according to T).

Define x < y of S to be y < x in T.

Interpretation of the axioms of S formally yields

i'. □(x < x).
ii'. (y < x □ z < y) □ z < x.
iii'. y < x   x < y   x = y.
iv'. (□x)(□y)(y < x   x = y).

These are obviously theorems of T.

Here is a more sophisticated example. PA = Peano Arithmetic is the first order theory with equality, using 0,S,+,•. The axioms: successor axioms, defining equations for +,•, and scheme of induction for all formulas in this language.

Now consider "finite set theory". This is a bit ambiguous: could mean either

ZFC without the axiom of infinity; i.e., or ZFC\I; or

ZFC with the axiom of infinity replaced by its negation; i.e., ZFC\I + ¬I.

THEOREM (well known). PA, ZFC\I, ZFC\I + ¬I are mutually interpretable.

PA in ZFC\I: nonnegative integers become finite von Neumann ordinals. Induction in PA gets translated to a consequence of foundation and separation.

ZFC\I + ¬I in PA: Sets of ZFC\I + ¬I, are coded by the natural numbers in PA – in an admittedly ad hoc manner.

The various axioms of ZFC\I + ¬I get translated into theorems of PA.

3. STARTLING FACT ABOUT INTERPRETATION POWER.

We begin with the observation that for any two S,T, if T is inconsistent (proves a sentence and its negation) then S is interpretable in T.

A very fundamental fact about interpretation power is that there is no greatest interpretation power – short of inconsistency.

THEOREM 3.1. (In ordinary predicate calculus with equality). Let S be a consistent recursively axiomatized theory in a finite language. ∃ a consistent finitely axiomatized conservative extension T of S which is not interpretable in S.

THEOREM 3.2. Let $S_1,...,S_k$ be consistent recursively axiomatized theories. There exists a consistent finitely axiomatized theory T, where
i) each S_i is interpretable in T;
ii) T is not interpretable in any of the S_i.

These can be proved using Gödel's second incompleteness theorem.

COMPARABILITY. Let S,T be recursively axiomatized theories. Then S is interpretable in T or T is interpretable in S?

Now there are plenty of quite interesting and natural examples of incomparability for finitely axiomatized theories that are rather weak. To avoid trivialities, we give an example of incomparability where there are only infinite models.

S is the theory of discrete linear orderings without endpoints.

T is the theory of dense linear orderings without endpoints.

We get plenty of incomparability arbitrarily high up:

THEOREM 3.3. Let S be a consistent recursively axiomatized theory. There exist consistent finitely axiomatized theories T_1,T_2, both in a single binary relation symbol, such that
i) S is provable in T_1,T_2;
ii) T_1 is not interpretable in T_2;
iii) T_2 is not interpretable in T_1.

BUT, are there examples of incomparability between natural theories that are metamathematically strong?

Metamathematically strong can be rephrased as "subject to the Gödel phenomena", and formally as "EFA is interpretable". Here EFA = exponential function arithmetic.

Natural theories are all recursively axiomatized, and in fact they are axiomatized by finitely many schemes. This includes finitely axiomatized theories.

STARTLING OBSERVATION. ***Any two natural theories S,T, known to interpret EFA, are known (with small numbers of exceptions) to have: S is interpretable in T or T is interpretable in S. The exceptions are believed to also have comparability.***

Because of this observation, there has emerged a rather large linearly ordered table of "interpretation powers" represented by natural formal systems. Generally, several natural formal systems may occupy the same position.

We call this growing table, the ***Interpretation Hierarchy***.

4. RESTATEMENT OF CONCEPT CALCULUS.

We restate what we said at the beginning, altered to take into account the Interpretation Hierarchy.

Concept Calculus develops with

a. An identification of a few related concepts from commonsense thinking. In the various developments, the choice of these concepts will vary greatly. In fact, all concepts from ordinary language are targets.

b. Formulation of a variety of fundamental principles involving these concepts. These various principles may have various degrees of plausibility, and may even be incompatible with each other. There may be no agreement among philosophers as to just which principles to accept. Concept Calculus is concerned only with logical structure.

c. Formulation of a variety of systems of such fundamental principles in b. These systems generally combine several such fundamental principles in some attractive way.

d. An identification of the position in the Interpretation Hierarchy of these resulting systems.

e. In particular, if the position is at or higher than that of ZFC, then we will generally have a finitary consistency proof of ZFC relative to the consistency of the system.

5. SOME DEVELOPMENTS IN CONCEPT CALCULUS.

We begin with the notions: better than (>), and much better than (>>). These are binary relations. This is an example of what we call concept amplification.

One can also view > and >> mereologically, as

x > y iff y is a “proper part of x”.

x >> y iff y is a “small proper part of x”.

BASIC. > is a linear ordering. $x >> y \rightarrow x > y$. $x >> y \wedge y > z \rightarrow x >> z$. $x > y \wedge y >> z \rightarrow x >> z$. $(\exists x)(x >> y,z)$. If x >> y then x >> some z minimally > y.

MINIMAL. There is nothing that is better than all minimal things.

EXISTENCE. Let x be a thing better than a given range of things. There is something that is better than the given range of things and the things that they are better than, and nothing else. Here we use L(>,>>) to present the range of things.

Existence is like fusion. Here the "range of things" is given by a first order formula in >,>>, with parameters allowed.

AMPLIFICATION. Let y > x be given, as well as a true statement about x, using the binary relation > and the unary relation >> x. The corresponding statement about x, using > and >> y, is also true.

THEOREM 5.1. Basic + Minimal + Existence + Amplification is mutually interpretable with ZFC. This is provable in EFA.

AMPLIFICATION (binary). Let y > x be given, as well as a true statement about x, using the binary relations > and z >> w >> x. The corresponding statement about x, using > and z >> w >> y, is also true.

AMPLIFIED LIMIT. There is something that is better than something, and also much better than everything it is better than.

Leads to much higher places in Interpretation Hierarchy than ZFC:

THEOREM 5.2. Basic + Minimal + Existence + Amplification (binary) interprets ZFC + "for all x □ □, x# exists" and is interpretable in ZFC + "there exists a measurable cardinal".

THEOREM 5.3. Add Amplified Limit: well above measurable cardinals. Below 'conentrating' measurable cardinals, in interpretation power.

I also considered a lot of theories based on time, and some on time and space. The simplest one to present involves a single varying quantity - where the time and quantity scale are the same.

The language has >,>>,F, where >,>> are binary relations, and F is a one place function.

F(x) is the value of the varying quantity at time x.

When thinking of time, >,>> is later than and much later than. When thinking of quantity, >,>> is greater than and much greater than.

BASIC. > is a linear ordering. x >> y x > y. x >> y > z x >> z. x > y >> z x >> z. (x)(x >> y,z). If x >> y then x >> some z minimally > y.

ARBITRARY BOUNDED RANGES. Every bounded range of values is the range of values over some bounded interval. Here we use L(>,>>,F) to present the bounded range of values.

AMPLIFICATION. Let y > x be given, as well as a true statement about x, using the binary relation > and the unary relation >> x. The corresponding statement about x, using > and >> y is also true.

This also lands at ZFC in the Interpretation Hierarchy. We can strengthen as before:

AMPLIFICATION (binary). Let y > x be given, as well as a true statement about x, using the binary relations > and z >> w >> x. The corresponding statement about x, using > and z >> w >> y, is also true.

AMPLIFIED LIMIT. There is something that is greater than something, and also much greater than everything it is greater than.

As before, these latter two principles push the interpretation power well into the large cardinal hierarchy.

There are versions with continuously ordered >, using equidistance of intervals.

PRINCIPLE OF PLENITUDE

From Wikipedia, Plenitude Principle.

The principle of plenitude asserts that everything that can happen will happen.

The historian of ideas Arthur Lovejoy was the first to discuss this philosophically important Principle explicitly, tracing it back to Aristotle, who said that no possibilities which remain eternally possible will go unrealized, then forward to Kant, via the following sequence of adherents:

Augustine of Hippo brought the Principle from Neo-Platonic thought into early Christian Theology.

St Anselm 's ontological arguments for God's existence used the Principle's implication that nature will become as complete as it possibly can be, to argue that existence is a 'perfection' in the sense of a completeness or fullness.

Thomas Aquinas's belief in God's plenitude conflicted with his belief that God had the power not to create everything that could be created. He chose to constrain and ultimately reject the Principle.

Giordano Bruno's insistence on an infinity of worlds was not based on the theories of Copernicus, or on observation, but on the Principle applied to God. His death may then be attributed to his conviction of its truth.

Leibniz believed that the best of all possible worlds would actualize every genuine possibility, and argued in *Théodicée* that this best of all possible worlds will contain all possibilities, with our finite experience of eternity giving no reason to dispute nature's perfection.

Kant believed in the Principle but not in its empirical verification, even in principle.

The Infinite monkey theorem and Kolmogorov's zero-one law of contemporary mathematics echo the Principle. It can also be seen as receiving belated support from certain radical directions in contemporary physics, specifically the many-worlds interpretation of quantum mechanics and the cornucopian speculations of Frank Tipler on the ultimate fate of the universe.

See Concept Calculus, Preprints, #53,
http://www.math.ohio-state.edu/%7Efriedman/manuscripts.html