# Ineffability and Reflection

Stewart Shapiro & Gabriel Uzquiano

OSU & Oxford

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# Ineffability

Two suggestive remarks:

> *If we now put forward the general hypothesis that every categorically determined domain can also be interpreted as a set in some way, i.e., can appear as an element of a normal domain, it follows that to each normal domain there is a higher domain with the same basis.*

(Zermelo, E., 1930, 1232)

> *Any time we try to capture the universe from what we posititvely possess (or can express) we fail the task and the characterization is satisfied by certain (large) sets.*

(Wang, H. 1974, 555)

# Plan

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2. Framework
3. Ineffability
4. The matter of consistency
5. Interpreting set theory in FZ
6. The predicament

# Overview

Motivation:

- Isolate the import of ineffability against a background of minimal presuppositions, i.e., extensionality
- Explore the prospects for a neo-Fregean foundation of set theory as based on an abstraction principle for extensions

We will develop a restriction of Frege's Basic Law V based on a purely logical characterization of ineffability.

# Overview

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# Framework

We work in a second-order language with $\mathrm{ext}$ as the only non-logical primitive.

$$\forall F \forall G(\mathrm{ext}(F) = \mathrm{ext}(G) \leftrightarrow \forall x(Fx \leftrightarrow Gx)) \qquad \text{(Basic Law V)}$$

- $xEy :_{def} \exists F(y = ext(F) \wedge Fx)$

Unfortunately:

- If $R = [x : \neg xEx]$ and $r = \mathrm{ext}(R)$, then:

$$rEr \leftrightarrow \neg rEr$$

Our plan now is to restrict the assignment of extensional extensions to a certain well-behaved concepts. We will identify the extension of all other concepts concepts, which we will just ignore from now on.

## Preliminaries:

The relativization of a sentence $\phi$ to a concept $F$ is the result of relativizing the quantifiers in $\phi$ to $F$, by which we mean, as usual, the result of replacing:

- $\exists x(...)$ with: $\exists x(Fx \wedge ...)$
- $\exists X(...)$ with: $\exists X(\forall x(Xx \rightarrow Fx) \wedge ...)$
- $\forall x(...)$ with: $\forall x(Fx \rightarrow ...)$
- $\forall X(...)$ with: $\forall X(\forall x(Xx \rightarrow Fx) \rightarrow ...)$

We will write $\phi^F$ for the relativization of $\phi$ to $F$. More definitions:

$F \sim G$ iff $F$ is equinumerous to $G$

$F \prec G$ iff there is a 1-1 function from $F$ into $G$ but not vice versa.

$F \preceq G$ iff $F \prec G$ or $F \sim G$.

First Pass:

Say that a concept $F$ is characterized by a sentence of pure second-order logic $\phi$ iff $\phi^F$ and for all $G$, $\phi^G$ only if $F \sim G$.

Say that a concept $F$ is characterizable iff $F$ is characterized by some sentence $\phi$ of pure second-order logic.

In the present context, Zermelo's suggestion would become the hypothesis that all characterizable concepts have extensional extensions. But since there are at most countably many characterizable concepts, this principle would only yield countably many extensional extensions.

## Second Pass:

Think of the size a concept $F$ must have for $\phi^F$ to be true as a lower bound a sentence $\phi$ sets on the domain, e.g., $\exists x \exists y (x \neq y)$ sets a lower bound of two on the domain since $[\exists x \exists y (x \neq y)]^F$ will be only be true if $F$ has at least two instances.

Likewise, if $\Phi^{PA}$ is the conjunction of the axioms of second-order Peano arithmetic, then $\Phi^{PA}$ sets a lower bound of $\aleph_0$ on the domain.

Say that a concept $F$ is bounded by a sentence of pure second-order logic $\phi$ iff $F$ is no larger than the lower bound set by $\phi$. Or, in other words, iff for every concept $G$, $\phi^G$ is true only if $F \preceq G$.

Say that a concept $F$ is bounded iff $F$ is bounded by some sentence of pure second-order logic. $F$ is unbounded iff $F$ is not bounded.

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Say that a concept $F$ is bounded iff $F$ is bounded by some sentence of pure second-order logic. $F$ is unbounded iff $F$ is not bounded.

We are interested in the thought that all and only bounded concepts have extensional extensions.

In other words, we are interested in the following hypothesis:

$$\forall F \forall G((F \text{ bounded } \wedge \ G \text{ bounded}) \rightarrow (\mathrm{ext}(F) = \mathrm{ext}(G) \leftrightarrow \forall x(Fx \leftrightarrow Gx)) \tag{RV}$$

But for now, we will have to content ourselves with bounded as a primitive predicate, since no sentence of pure second-order logic expresses the fact that $F$ is bounded on pain of contradiction.

Since bounded must remain primitive, we will have to content ourselves with an axiom schema:

$$F \text{ unbounded} \rightarrow ((F \preceq H \wedge \phi^H) \rightarrow \exists G(G \prec H \wedge \phi^G)) \qquad \text{(RF)}$$

We then obtain a sufficient condition for $F$ to be bounded:

- $(F \preceq H \wedge \phi^H \wedge \forall G(G \prec H \rightarrow \neg\phi^G)) \rightarrow F$ bounded

In particular:

- $(\phi^F \wedge \forall G(G \prec F \rightarrow \neg\phi^G)) \rightarrow F$ bounded

Some consequences of (RF) + (RV):

- $\exists F(F$ unbounded$)$
- $V$ unbounded (where $V$ is the universal concept)
- $\phi^V \rightarrow \exists G(G \prec V \wedge \phi^G)$

Assuming a well-ordering principle, (RF) would give us:

- $\phi^F \rightarrow \exists G(G \text{ bounded} \wedge \phi^G)$

And, in combination with, (RV):

- $\phi^F \rightarrow \exists x \exists G(x = \mathrm{ext}(G) \wedge \phi^x)$

where $\phi^x$ is the relativization of $\phi$ to the members of $x$.

This is close to Bernays' reflection principle but not quite.

However, (RF) gives only partial expression to the claim that a concept is $F$ bounded iff $F$ is bounded by some sentence. In particular, there is no guarantee that $F$ is bounded *only if* $F$ is bounded by some sentence. We could address this limitation in at least two ways:

(1) Supplement (RV) + (RF) with further axioms giving voice to the assumption that being bounded is largely a matter of size. For example:

$$(H \text{ bounded } \wedge\ F \preceq H) \rightarrow F \text{ bounded}$$

(2) Ascent to a third-order language in which:

$$F \text{ bounded by a sentence } \phi \text{ of pure second-order logic}$$

is given an explicit definition. (More on this later.)

# The matter of consistency

Let FZ (for Frege-Zermelo) be the theory of extensions whose axioms are:

$$\begin{aligned} &\forall F \forall G((F \text{ bounded } \wedge \ G \text{ bounded}) \rightarrow \\ &(\mathrm{ext}(F) = \mathrm{ext}(G) \leftrightarrow \forall x(Fx \leftrightarrow Gx)) \end{aligned} \tag{1}$$

$$F \text{ unbounded} \rightarrow ((F \preceq H \wedge \phi^H) \rightarrow \exists G(G \prec H \wedge \phi^G)) \tag{2}$$

## Remark (ZFC)

*FZ is consistent*

For each sentence of pure second-order logic $\Phi$, let $f(\langle\Phi\rangle)$ be the least $\lambda$ such that $\Phi^\lambda$ is true, if there is such a $\lambda$, and 0 otherwise.

$f$ is a function from sentences of pure second-order logic into cardinals. Let $\kappa$ be $\bigcup ran\ f$. Then:

- $\kappa$ is a limit cardinal of cofinality $\omega$

To obtain a model of FZ, we have to produce a set $M$ such that:

- $\kappa \subseteq M$
- If $X \subseteq M$ and $|X| < \kappa$, then $X \in M$

($\kappa$ will be the least unbounded size of the model.)
Define a function $\pi_\alpha$ by transfinite recursion on the ordinals $\leq \kappa^+$:

$$\begin{aligned} \pi_0 &= \kappa \\ \pi_{\alpha+1} &= \pi_\alpha \cup \{x : x \subseteq \pi_\alpha \wedge |x| < \kappa\} \\ \pi_\lambda &= \bigcup_{\beta<\lambda} \pi_\beta \end{aligned}$$

Since $\kappa^+$ is a regular cardinal, $\pi_{\kappa^+}$ is a fixed-point of $\pi$ that will serve as a domain $M$ for our model. The extension of a subset $X$ of $M$ of size $< \kappa$ will be $X$ itself, which is guaranteed to be a member of $M$.

# Interpreting pure set theory in FZ

Definitions:

$$x \in y :_{def} \quad \exists F(y = \mathrm{ext}(F) \wedge Fx)$$

Following (Boolos 1989), call a concept $F$ closed iff

$$\forall y((\exists G(y = \mathrm{ext}(G) \wedge \forall z(z \in y \rightarrow Fz)) \rightarrow Fy)$$

In other words, $F$ is closed if, whenever it holds of the members of an extension, it holds of that extension.

$$Set(x) :_{def} \quad \forall F(F \text{ closed} \rightarrow Fx)$$

## Axioms of Set Existence:

- Null Set: $\exists x(Set(x) \wedge \forall y \, \neg y \in x)$

  $[x : x \neq x]$ is bounded by the sentence $\forall x(x \neq x)$. Therefore, $\mathrm{ext}([x : x \neq x])$ is the extension of a bounded concept and hence a set.

- Singletons: $\forall x \exists y(Set(y) \wedge \forall z(z \in y \leftrightarrow z = x)$

  If $a$ is a pure set, then $[x : x = a]$ is bounded by $\exists x(x = x)$. Its extension is a pure set, i.e., the singleton of $a$.

- Pairs: $\forall x \forall y \exists z(Set(z) \wedge \forall w(w \in z \leftrightarrow (w = x \vee w = y))$

  If $a$ and $b$ are pure sets, $[x : x = a \vee x = b]$ is bounded by $\exists x \exists y(x \neq y)$. Its extension is the pair set of $a$ and $b$.

- Infinity:

  Let $F$ be the least concept under which the null set falls which is closed under singletons. Let $\Phi^{\text{DedInf}}$ be the sentence:

  $$\exists f(\forall x \forall y(fx = fy \rightarrow x = y) \wedge \exists x \forall y(x \neq fy))$$

  $F$ is bounded by $\Phi^{\text{DedInf}}$, and its extension is an infinite set.

Thus far, we have only used the fact that a sufficient condition for $F$ to be bounded is for $F$ to be bounded by some sentence, which is what (RF) manages to express. In order to prove power set, separation and replacement, we would need the converse.

However, it is not difficult to check that they are each satisfied by extensions of concepts bounded by second-order sentences.

Alternatively, we could prove separation and replacement if we assumed the axiom:

$$(H \text{ bounded } \wedge \ F \preceq H) \rightarrow F \text{ bounded}$$

- Foundation: See (Boolos 1989).
- Choice: Whether or not Choice is a consequence of FZ will largely depend on whether we adopt a form of choice as a logical principle of the background logic.

## An Important Omission:

Union: Union fails in the model given above, since $\kappa$, the least unbounded size in our model, has cofinality $\omega$.

At most, we have a restricted form of union:

> *Restricted Union: If some sentence $\phi$ of pure second-order logic sets an upper bound for every concept $G$ whose extension is in $\mathrm{ext}(F)$, then $\mathrm{ext}(F)$ has a union, i.e., $[x : \exists y(y \in \mathrm{ext}(F) \wedge x \in y)]$ has an extensional extension.*

However:

FZ yields the existence of $\omega$, $\mathcal{P}\omega$, $\mathcal{P}\mathcal{P}\omega$, etc. With replacement we are able to form $V_{\omega+\omega}$ and well beyond. This is enough set theory to recover much of ordinary mathematics.

Now:

Recall the complaint that (RF) gives only partial expression to the claim that a concept is bounded if and only if it is bounded by some sentence of pure second-order logic.

If we had the means to give complete expression to this biconditional, then we would be able to derive separation, replacement and power set.

# The Predicament

One way to improve on (RF) would be to ascent to a third-order language. Let $\Im$ be a third-order variable ranging over concepts of concepts. There is a third-order formula of the language of pure third-order logic, $DEF(\Im)$ that expresses the fact that for some sentence of pure second-order logic $\phi$, for all $F$:

$\Im(F)$ *if and only if* $\phi^F$ *is true.*

Then we can turn (RF), which used to be an axiom schema, into an explicit definition:

$$F \text{ unbounded} \leftrightarrow \forall \Im(DEF(\Im) \wedge \exists H(F \preceq H \wedge \Im H) \rightarrow \exists G(G \prec H \wedge \Im G))) \tag{RF3}$$

Or if you prefer:

$$F \text{ bounded} \leftrightarrow \exists \Im (DEF(\Im) \wedge \exists H(F \preceq H \wedge \Im H) \wedge \forall G(G \prec H \rightarrow \neg \Im G))) \tag{RF3}$$

With an explicit definition of bounded in place, we may replace our three axioms by a single version of (RV) in which "$F$ bounded" and "$G$ bounded" are suitably replaced with their definientia in accordance to (RF3):

$$\forall F \forall G((F \text{ bounded } \wedge \ G \text{ bounded}) \rightarrow (\mathrm{ext}(F) = \mathrm{ext}(G) \leftrightarrow \forall x(Fx \leftrightarrow Gx)) \tag{RV3}$$

Now: each instance of the (RF) schema is now a consequence of (RV3), which is a single axiom from which we derive all of of ZF with the exception of union.

However:

Will (RV3) give complete expression to the inchoate thought that the universe is ineffable? Not if by bounded we mean bounded by a sentence of pure third-order logic. When we moved to a third-order language, we thereby moved the goal post and bounded by a sentence of pure second-order logic no longer means bounded by a sentence of pure logic.

What happens when we understand bounded as bounded by sentences of pure third-order logic? We may offer a third-order version of (RF) in which we allow substitution by arbitrary third-order sentences and content ourselves with the schema. Or we may ascent to a fourth-order language and replace (RF3) by an explicit definition of bounded by a sentence of third-order logic. And we may then replace (RV) by a single axiom (RV4).

Will (RV4) give complete expression to the inchoate thought that the universe is ineffable? Not if by bounded, we mean bounded by a sentence of pure fourth-order logic. Etc.

Notice, however, that the consistency proof given above can be adapted to show that ZFC proves the consistency of each of (RV3), (RV4), etc.