IS LOGIC AXIOMATIZABLE?

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I. INTRODUCTION

No. Logic is NOT axiomatizable.

The RECEIVED VIEW:

RV1. *Mathematics* is not axiomatizable.
Arithmetic is not axiomatizable.
RV2. *Logic* is axiomatizable.
Elementary logic is axiomatizable.
(Logic *must* be axiomatizable.)

NOTES:

1. Two notions of axiomatization
   Dedekind-Peano 'axiomatization' of arithmetic
   2^{nd}-order language: 2^{nd}-order consequences
2. Higher-order logics (e.g., second-order logic), unlike elementary logic, are not axiomatizable.
3. The Received View:
   RV3. The (full) second-order logic is *not* logic in the proper sense
   It is a theory of sets (or classes) in disguise.
   Second-order quantifiers are quantifiers over sets (or classes).
4. Dissenters of Two Kinds
   A. Higher-order logics (e.g., second-order logic) are logics in the full-blown sense.
   B. What to call logic, and what not, is purely arbitrary. It's a mere verbal issue.

Logic itself is not axiomatizable.

1. Logic must include the logical relations that pertain to *plural constructions* of natural languages as well.
2. Those logical relations cannot be captured by axiomatic systems.
2.1. The logic of plurals is non-compact and, thus, non-axiomatizable.

Argument 1:

C_1 admires C_2. C_2 admires C_3. .... C_n admires C_{n+1}. ....

:. There are some things each one of which admires one of them.

II. PLURALS AND THEIR LOGIC

PLURAL CONSTRUCTIONS
versus SINGULAR CONSTRUCTIONS

	PLURALS		SINGULARS
P1	John and Carol *are children*.	S1	John is a child.
P2	John and Carol *lift* Bob.	S2	John *lifts* Bob.
P3	Mary's children *are* John and Carol.	S3	John *is* Mary's only child.
P4	There are some *children* who *lift* Bob.	S4	There is a *child* who *lifts* Bob.
P5	Mary's children *lift* Bob.	S5	Mary's only child *lifts* Bob.

LOGIC THAT PERTAINS TO PLURALS

[P1] and [P2] logically imply [P4].
[P2] and [P3] logically imply [P5].
[P1] logically implies [S1].

[P2] does not logically imply [S2].
[P4] does not logically imply [S4].

III. NON-COMPACTNESS

Def. 1 (compactness):

A. A sentence *S* is *non-compact* (in a language *L*) if and only if *S* is logically implied by some sentences (in *L*) while not being logically implied by any finitely many sentences among them.

B. The logic of a language *L* is *non-compact*, if *L* has a non-compact sentence; and compact otherwise.

Thesis NC: Some natural language sentences are NON-COMPACT.

[a] There are some things each one of which admires one of them.

[b] *The Geach-Kaplan Sentence:* Some critics admire only one another. (There are some critics each one of whom admires nothing but another one of them.)

[c] There are some weird things that include (i) Jack and (ii) any friend of any one of them. (There are some weird things of which Jack is one, and of which any friend of any one of them is one.) [Note: "*to* include" is the converse of "*to* be some of" (or "*to* be among")]

Argument 1:

C_1 admires C_2. C_2 admires C_3. .... C_n admires C_{n+1}. ....

:. There are some things each one of which admires one of them.

Argument 2:

C_1 is a critic. C_2 is a critic. .... C_n is a critic. ....
C_1 is not C_2. C_2 is not C_3. .... C_n is not C_{n+1}. ....
C_1 admires only C_2. C_2 admires only C_3. .... C_n admires only C_{n+1}. ....

:. There are some critics who admire only one another.

Argument 3:

Jack admires something.
<u>Anything admired by</u> *Jack* admires something.
<u>Anything admired by</u> *anything admired by Jack* admires something.
.
.
.

:. There are some things each one of which admires one of them.

Argument 4:

Jack is weird.
<u>Any friend of</u> *Jack* is weird.
<u>Any friend of</u> *any friend of Jack* is weird.
.
.
.

:. There are some weird things that include (i) Jack and (ii) any friend of any one of them.

IV. NON-AXIOMATIZABILITY

Def. 2. (*Strong) Axiomatizability*:
The logic of a language *L* is (*strongly*) *axiomatizable* if and only if there is a deductive system, *D*, that is complete and sound with respect to the logical consequence among sentences of *L* (that is, some sentences of *L* logically imply a sentence of *L* if and only if the latter is deducible from the former in *D*).

If the logic of a language is (strongly) axiomatizable, the logic is compact.
The logic of a language that includes basic plurals (e.g., [a]-[c]) are non-compact.
So the logic of a language that includes basic plurals (e.g., [a]-[c]) are not axiomatizable.

ASIDE: The logic of plurals is not even *weakly axiomatizable*.

Kaplan's proof of the inexpressibility of [b] in elementary languages
(See Boolos (1984))
Categorical 'axiomatization' of arithmetic
Plural cousin of the second-order induction principle
"Some positive natural numbers are successors only of one another."

Def. 3. (*Weak) Axiomatizability*:
The logic of a language *L* is (*weakly*) *axiomatizable* if and only if there is a deductive system, *D*, that is complete and sound with respect to the logical truths of *L* (that is, a sentence of *L* is a logical truth if and only if it is a theorem in *D*).

V. ADDITIONAL REMARKS

1. ACCOUNTS OF LOGIC
versus LOGICAL DATA

Accounts/theories of the logic of plurals
1. Traditional Approach
Plurals are more or less devices for abbreviating singulars.
Reducing plurals to singulars
Paraphrasing plurals into singular regimented languages
Elementary languages
Higher-order languages built on elementary languages
2. Non-reductionist Approach
Plurals are not devices for abbreviating singulars.
Plurals have distinct semantic functions.
Devices for talking about many things as such.
One can say more things using plurals than using only singulars.
Regimented languages used to account for the logic of plurals must have refinements of natural language plurals.

My argument for the non-compactness does not rest on my theory of plurals.
Appeals directly to Intuitive logical data

2. Limitations of Elementary Languages

Inexpressibility of [a]-[c] in elementary languages. (Cf. David Kaplan on [b])

No elementary language sentence is logically equivalent to [a]; elementary logic is compact.

In particular, [a] cannot be paraphrased by an elementary language sentence (e.g., [a*]) that invokes sets (or classes or other composite objects).

[a*] There is something that has a member, and each member of which admires a member of it the former.

[1] Ezra admires Thomas, and Thomas admires Ezra.
[2] Ezra admires one of Ezra and Thomas, and Thomas admires one of Ezra and Thomas.
[3] Each one of Ezra and Thomas admire one of Thomas and Ezra.
[4] Each one of Ezra and Thomas admire one of them.
[a] There are some things each one of which admires one of them.

3. The argument does NOT assume that plural constructions are higher-order constructions.

We can see the logic of [a] without assuming that [a] can be paraphrased into second-order languages in order to appeal to the (full) second-order logic.

Aside 1: Cf. Kaplan's proof of the inexpressibility of [b] in elementary languages.

"Some positive natural numbers are successors only of one another."

Aside 2: Some plural constructions of natural languages cannot be paraphrased even into the usual (i.e., singular) higher-order languages.

E.g., "Those who wrote *Principia Mathematica* cooperated (to lift a piano)."

4. Whether logic is compact/axiomatizable is *not* a mere verbal issue.

Logic must cover plurals as well as singulars.

The logic of languages that contain basic plurals must be non-compact.

Aside: The logic of plurals does not include mathematics (e.g., arithmetic or set theory).

5. Comparison with Tarski's ω-consequence example

Tarski's argument:

1 is weird. *2* is weird.... *N* is weird....

:. Every *number* is weird.

The argument (as it stands) is not *logically* valid.

Definitions of the italicized expressions: "number", and "1", "2", etc.

But it is related to a logically valid argument, such as *Argument 4*.

6. The use of numerical subscripts in the sentences involved in *Argument 1* is not essential.[1] See *Arguments 3 & 4*.

7. We can specify the infinitely many premisses of *Argument 3* (or *4*) without invoking natural numbers or using set-theoretic notions.

> The premisses of the argument *include* (1) "Jack admires something", and (2) any sentence that results from prefixing "anything admired by" to any one of them; and they are some of (i.e., *included by*) any sentences that include (1) "Jack admires something", and (2) any sentence that results from prefixing "anything admired by" to any one of them.

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[1] Some might hold that *Argument 1* is not logically valid in the strict sense but, so to speak, 'arithmetically valid': because its premisses involve numerals, understanding of natural numbers is required for one to see that its conclusion follows from the premisses.

APPENDIX 1: DATA ON LOGIC

SINGULAR

John is a child, and Carol is a child.
:. John is a child.

Every horse is an animal. Every animal can fly.
:. Every horse can fly.

Every horse is an animal.
:. Every head of a horse is a head of an animal.

PLURAL

The horses in this room are mammals in this room. The mammals in this room can fly.
:. The horses in this room can fly.

> The horses in this room are mammals in this room.
> The mammals in this room can lift Bob (if they cooperate).
> :. The horses in this room can lift Bob (if they cooperate). [*INVALID*]

John and Carol are children. :: John is a child, and Carol is a child.

> John and Carol can lift Bob. :. John can lift Bob. [*INVALID*]

Ezra and Thomas are critics who admire only each other.
:. Some critics admire only one another.

Russell and Whitehead cooperate. They are those who wrote *PM*.
:. Those who wrote *PM* cooperate.

Russell and Whitehead cooperate.
Russell and Whitehead are philosophers who coauthor *PM*.
:. There are some philosophers who cooperate and coauthor *PM*.

Cicero is one of John and Carol. :: Cicero is John or Cicero is Carol.

Cicero is one of those who can fly. :: Cicero can fly.

> Cicero is one of those who wrote *PM*. :. Cicero wrote *PM*. [INVALID]

There is a logician who lives in Stanford.
:. There are some logicians who live in Stanford such that every logician who lives in Stanford is one of them.

APPENDIX II: AN ACCOUNT OF PLURALS AND THEIR LOGIC (Sketch)

1. DISTINCT FUNCTION OF PLURALS

Plurals are *essentially* devices for talking about *MANY* things (‘at a time’).

Cf. *Singulars* are devices for talking about *ONE* thing (‘at a time’).

	Plural	Singular
Terms	“John and Carol” “Mary’s children” “they”	“John”, “Carol”, “Bob” “Mary’s only child” “he”, “she”, “it”
Predicates or Predicate Phrases	“are children” “lift” (or “*to* lift) “are” (or “*to* be”) “is one of”	“is a child” “admires” “is” (or “is identical with”)
Quantifiers or Quantifier Phrases	“There are some”	“There is a”

A. A typical plural term refers to (or denote) many things (‘at once’).

“John and Carol” refers to the two children: John and Carol.

B. A plural predicate indicates a *plural attribute*.

“*to* lift Bob” indicates a property that admits two or more things ‘at once’.

“*to* be two humans” indicates the property of *being two humans*.

C. Plural Quantifiers are also devices for generalization

Roughly, “There are some” is to plural terms (e.g., “they”) what “There is a” is to singular terms (e.g., “it”).

NOTE: More precisely, plurals rather relate to *one or many* things.

A plural term (e.g., “Cicero and Tully”) may refer to one thing only.

A plural predicate typically has a neutral argument place.

“some” is interchangeable with “some one or more” rather than “two or more”.

2. PLURALS ARE NOT DEVICES FOR ABBREVIATING SINGULARS.

A. Plurals cannot be reduced to singulars.

Some plurals do not have singular equivalents.

“Those who wrote *PM* cooperated (to carry Bob)”

B. Plurals cannot be paraphrased into the ususal regimented languages.

Singularity of elementary languages and their higher-order extensions

Expressions of elementary languages:

refinements of basic singular constructions

Singular Constants, Variables, Predicates, Quantifiers

Higher-Order Expressions built on elementary language expressions

2^{nd} order variables match elementary language predicates.

3. REGIMENTING PLURALS

plural languages: 1st order extensions of elementary languages

refinements of plural terms as non-predicable expressions

plural predicates and quantifiers

Primitives of Plural Languages

	SINGULAR	PLURAL
TERMS	*Constants*: "c", "j", "m" *Variables*: "*x*", "*y*", "*z*"	*Variables*: "*xs*", "*ys*", "*zs*"
PREDICATES *1-place*	"C" ("is-a-child")	"$\mathbf{C_o}$" ("**cooperate**")
2-place	"A" ("admires"), "ε", "<"	"**L**" ("**lift**"), "**W**" ("**write**")
LOGICAL	"="	"H" ("**is-one-of**")
Quantifiers	"∃" ("There-is-something ... such-that")	"Σ" ("there-are-some-things ... such-that")

Defined Expressions:

1. Universal Quantifiers: "Π", "∀"
2. The sameness predicate "≡" ("*to* be")
3. Neutral Expansions:
   "is-a-child" to "*to* be a-child/children"
   xs are-children iff everything that is-one-of *xs* is-a-child.
4. Definite Descriptions
   Singular: "the president of the USA"
   Plural: [a] "the residents of London"; [b] "those who wrote *PM*"
5. Conjunctive Plural Terms
   "John and Carol", "Cicero and those who wrote *PM*"

4. PLURAL LOGIC

Characterize the Logic of Plural Languages