

First-Order Gödel Logics

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Abstract

First-order Gödel logics are a family of finite- or infinite-valued logics where the sets of truth values V are closed subsets of $[0, 1]$ containing both 0 and 1. Different such sets V in general determine different Gödel logics \mathbf{G}_V (sets of those formulas which evaluate to 1 in every interpretation into V). It is shown that \mathbf{G}_V is axiomatizable iff V is finite, V is uncountable with 0 isolated in V , or every neighborhood of 0 in V is uncountable. Complete axiomatizations for each of these cases are given. The r.e. prenex, negation-free, and existential fragments of all first-order Gödel logics are also characterized.

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1 Introduction

1.1 Motivation

The logics we investigate in this paper, first-order Gödel logics, can be characterized in a rough-and-ready way as follows: The language is a standard first-order language. The logics are many-valued, and the sets of truth values considered are closed subsets of $[0, 1]$ which contain both 0 and 1. 1 is the “designated value,” i.e., a formula is valid if it receives the value 1 in every interpretation. The truth functions of conjunction and disjunction are minimum and maximum, respectively, and quantifiers are defined by infimum and supremum over subsets of the set of truth values. The characteristic operator of Gödel logics, the Gödel conditional, is defined by $a \rightarrow b = 1$ if $a \leq b$ and $= b$ if $a > b$. Because the truth values are ordered (indeed, in many cases, densely ordered), the semantics of Gödel logics is suitable for formalizing *comparisons*. It is related in this respect to a more widely known many-valued logic, Łukasiewicz (or “fuzzy”) logic—although the truth function of the Łukasiewicz conditional is defined not just using comparison, but also addition. In contrast to Łukasiewicz logic, which might be considered a logic of *absolute* or *metric comparison*, Gödel logics are logics of *relative comparison*. This alone makes Gödel logics an interesting subject for logical investigations.

There are other reasons why the study of Gödel logics is important. As noted, Gödel logics are related to other many-valued logics of recognized importance. Indeed, Gödel logic is one of the three basic *t*-norm based logics which have received increasing attention in the last 15 or so years (the others are Łukasiewicz and product logic; see Hájek 1998). Yet Gödel logic is also closely related to intuitionistic logic: it is the logic of linearly-ordered Heyting algebras. In the propositional case, infinite-valued Gödel logic can be axiomatized by the intuitionistic propositional calculus extended by the axiom schema $(A \rightarrow B) \vee (B \rightarrow A)$. This connection extends also to Kripke semantics for intuitionistic logic: Gödel logics can also be characterized as logics of (classes of) linearly ordered and countable intuitionistic Kripke structures with constant domains (Beckmann and Preining, 200?).

One of the surprising facts about Gödel logics is that whereas there is only one infinite-valued propositional Gödel logic, there are infinitely many different infinite-valued first-order Gödel logics depending on the choice of the set of truth values. This is also the case when one considers the propositional entailment relation (Baaz and Zach, 1998), and likewise when the language is extended to include quantification over propositions (Baaz and Veith, 2000, Theorem 4). For both quantified propositional and first-order Gödel logics, different sets of truth values with different order-theoretic properties in general result in different sets of valid formulas. Hence it is necessary to consider truth value sets other than the standard unit interval $[0, 1]$.

Our focus in this paper is on the recursive axiomatizability of entailment relations in first-order Gödel logics, where entailment is defined as: $\Gamma \vDash A$ if for every interpretation \mathfrak{I} , $\inf\{\mathfrak{I}(B) : B \in \Gamma\} \leq \mathfrak{I}(A)$. Entailment in infinite-valued logics may fail to be axiomatizable by a deduction system simply because the logic is not compact, as is the case in propositional infinite-valued Łukasiewicz logic. If the entailment relationship is not compact, it is still of interest to determine if the corresponding set of validities is axiomatizable (as is the case for propositional Łukasiewicz logic).

Scarpellini (1962) has shown that the set of validities of infinite-valued first-order Łukasiewicz logic is not axiomatizable, and this result can be extended to almost all linearly ordered infinite-valued logics. It is thus surprising that some infinite-valued Gödel logics are axiomatizable. Our main aim in this paper is to characterize those sets of truth values which give rise to axiomatizable Gödel logics, and those whose sets of validities are not recursively enumerable (r.e.). We show that a set V of truth values determines an axiomatizable first-order Gödel logic if, and only if, V is finite, V is uncountable and 0 is isolated, or every neighborhood of 0 in V is uncountable. We give strong completeness results, which also establish that the axiomatizability extends to entailment and not just to the set of validities, and that the logics concerned are thus compact. The kinds of truth value sets resulting in axiomatizable logics moreover determine different sets of validities (and entailment relations): the finite-valued Gödel logics \mathbf{G}_n , the logic \mathbf{G}_0 , and the “standard” infinite-valued Gödel logic $\mathbf{G}_{\mathbb{R}}$ (as defined on the truth value set $[0, 1]$).

1.2 History of Gödel logics

Gödel logics are one of the oldest families of many-valued logics. Propositional finite-valued Gödel logics were introduced by Gödel (1933) to show that intuitionistic logic does not have a characteristic finite matrix. They provide the first examples of intermediate logics (intermediate, that is, in strength between classical and intuitionistic logics). Dummett (1959) was the first to study infinite-valued propositional Gödel logics, axiomatizing the set of tautologies over infinite truth-value sets by intuitionistic logic extended by the linearity axiom $(A \rightarrow B) \vee (B \rightarrow A)$. Hence, infinite-valued propositional Gödel logic is also sometimes called Gödel-Dummett logic or Dummett’s **LC**. In terms of Kripke semantics, the characteristic linearity axiom picks out those accessibility relations which are linear orders. The entailment relation in propositional Gödel logics was investigated by Baaz and Zach (1998) and Gödel logics with quantifiers over propositions by Baaz et al. (2000).

Standard first-order Gödel logic $\mathbf{G}_{\mathbb{R}}$ —the one based on the full interval $[0, 1]$ —has been discovered and studied by several people independently. Alfred Horn was probably the first: He discussed this logic under the name *logic with truth values in a linearly ordered Heyting algebra* (Horn, 1969), and gave an axiomatization and the first completeness proof. Takeuti and Titani (1984) called $\mathbf{G}_{\mathbb{R}}$ *intuitionis-*

tic fuzzy logic and gave a sequent calculus axiomatization for which they proved completeness. This system incorporates the density rule

$$\frac{\Gamma \vdash A \vee (C \rightarrow p) \vee (p \rightarrow B)}{\Gamma \vdash A \vee (C \rightarrow B)}$$

(where p is any propositional variable not occurring in the lower sequent.) The rule is redundant for an axiomatization of $\mathbf{G}_{\mathbb{R}}$, as was shown by Takano (1987), who gave a streamlined completeness proof of Takeuti-Titani's system without the rule. A syntactical proof of the elimination of the density rule was later given by Baaz and Zach (2000). Other proof-theoretic investigations of Gödel logics can be found in Baaz and Ciabattoni (2002) and Baaz et al. (2003a). The density rule is nevertheless interesting: It forces the truth value set to be dense in itself (in the sense that, if the truth value set is not dense in itself, the rule does not preserve validity). This contrasts with the expressive power of formulas: no formula is valid only for truth value sets which are dense in themselves.

First-order Gödel logics other than $\mathbf{G}_{\mathbb{R}}$ were first considered in Baaz et al. (1996b), where it was shown that \mathbf{G}_{\downarrow} , based on the truth value set $V_{\downarrow} = \{1/k : k \in \mathbb{N} \setminus \{0\}\} \cup \{0\}$ is not r.e. Hájek (2005) has recently improved this result, and showed that not only is the set of validities not r.e., it is not even arithmetical. Hájek also showed that the Gödel logic \mathbf{G}_{\uparrow} based on $V_{\uparrow} = \{1 - 1/k : k \in \mathbb{N} \setminus \{0\}\} \cup \{1\}$ is Π_2 -complete. Results preliminary to the results of the present paper were reported in Baaz et al. (2003b) and Preining (2002, 2003).

1.3 Overview of the results

We begin with a preliminary discussion of the syntax and semantics of Gödel logics, including a discussion of some of the more interesting special cases of first-order Gödel logics and their relationships (Section 2). In Section 3, we present some relevant results regarding the topology of truth-value sets.

The main results of the paper are contained in Sections 4–6. We provide a complete classification of the axiomatizability of first order Gödel logics. The main results are that a logic based on a truth value set V is axiomatizable if and only if

- (1) V is finite (Section 6), or
- (2) V is uncountable and 0 is contained in the perfect kernel (Section 5.1), or
- (3) V is uncountable and 0 is isolated (Section 5.2).

In all other cases, i.e., for logics with countable truth value set (Section 4) and those where there is a countable neighborhood of 0 and 0 is not isolated (Section 5.3), the respective logics are not r.e.

In Section 7, we investigate the complexity of fragments of first-order Gödel logic, specifically, the prenex fragments (Section 7.1), the \forall -free fragments (Section 7.2), and the \perp -free fragments (Section 7.3). We show that the prenex fragment of a Gödel logic is r.e. if and only if the truth value set is finite or uncountable. This means that there are truth-value sets where the prenex fragment of the corresponding logic is r.e. even though the full logic is not. Moreover, the prenex fragments of all uncountable Gödel logics coincide. The same is also the case for \perp -free and \forall -free fragments.

2 Preliminaries

2.1 Syntax and Semantics

In the following we fix a standard first-order language \mathcal{L} with finitely or countably many predicate symbols P and finitely or countably many function symbols f for every finite arity k . In addition to the two quantifiers \forall and \exists we use the connectives \vee , \wedge , \rightarrow and the constant \perp (for ‘false’); negation is introduced as an abbreviation: we let $\neg A \equiv (A \rightarrow \perp)$. For convenience, we also define $\top \equiv \perp \rightarrow \perp$.

Gödel logics are usually defined using the single truth value set $[0, 1]$. For propositional logic, any choice of an infinite subset of $[0, 1]$ leads to the same propositional logic (set of tautologies). In the first order case, where quantifiers will be interpreted as infima and suprema, a closed subset of $[0, 1]$ is necessary.

Definition 2.1 (Gödel set). A *Gödel set* is a closed set $V \subseteq [0, 1]$ which contains 0 and 1.

The semantics of Gödel logics, with respect to a fixed Gödel set as set of truth values and a fixed language \mathcal{L} of predicate logic, is defined using the extended language \mathcal{L}^U , where U is the universe of the interpretation \mathfrak{I} . \mathcal{L}^U is \mathcal{L} extended with constant symbols for each element of U .

Definition 2.2 (Semantics of Gödel logic). Let V be a Gödel set. An *interpretation* \mathfrak{I} into V , or a *V-interpretation*, consists of

- (1) a nonempty set $U = U^{\mathfrak{I}}$, the ‘universe’ of \mathfrak{I} ,
- (2) for each k -ary predicate symbol P , a function $P^{\mathfrak{I}}: U^k \rightarrow V$,
- (3) for each k -ary function symbol f , a function $f^{\mathfrak{I}}: U^k \rightarrow U$.
- (4) for each variable v , a value $v^{\mathfrak{I}} \in U$.

Given an interpretation \mathfrak{I} , we can naturally define a value $t^{\mathfrak{I}}$ for any term t and a truth value $\mathfrak{I}(A)$ for any formula A of \mathcal{L}^U . For a term $t = f(u_1, \dots, u_k)$ we define $\mathfrak{I}(t) = f^{\mathfrak{I}}(u_1^{\mathfrak{I}}, \dots, u_k^{\mathfrak{I}})$. For atomic formulas $A \equiv P(t_1, \dots, t_n)$, we define $\mathfrak{I}(A) =$

$P^{\mathfrak{J}}(t_1^{\mathfrak{J}}, \dots, t_n^{\mathfrak{J}})$. For composite formulas A we define $\mathfrak{J}(A)$ by:

$$\mathfrak{J}(\perp) = 0 \quad (1)$$

$$\mathfrak{J}(A \wedge B) = \min(\mathfrak{J}(A), \mathfrak{J}(B)) \quad (2)$$

$$\mathfrak{J}(A \vee B) = \max(\mathfrak{J}(A), \mathfrak{J}(B)) \quad (3)$$

$$\mathfrak{J}(A \rightarrow B) = \begin{cases} 1 & \mathfrak{J}(A) \leq \mathfrak{J}(B) \\ \mathfrak{J}(B) & \mathfrak{J}(A) > \mathfrak{J}(B) \end{cases} \quad (4)$$

$$\mathfrak{J}(\neg A) = \begin{cases} 1 & \mathfrak{J}(A) > 0 \\ 0 & \mathfrak{J}(A) = 0 \end{cases} \quad (5)$$

$$\mathfrak{J}(\forall x A(x)) = \inf\{\mathfrak{J}(A(u)) : u \in U\} \quad (6)$$

$$\mathfrak{J}(\exists x A(x)) = \sup\{\mathfrak{J}(A(u)) : u \in U\} \quad (7)$$

(Here we use the fact that every Gödel set V is a *closed* subset of $[0, 1]$ in order to be able to interpret \forall and \exists as \inf and \sup in V . (5) is a consequence of the definition of $\neg A \equiv A \rightarrow \perp$.)

If $\mathfrak{J}(A) = 1$, we say that \mathfrak{J} *satisfies* A , and write $\mathfrak{J} \models A$. If $\mathfrak{J}(A) = 1$ for every V -interpretation \mathfrak{J} , we say A is *valid* in \mathbf{G}_V and write $\mathbf{G}_V \models A$.

If Γ is a set of sentences, we define $\mathfrak{J}(\Gamma) = \inf\{\mathfrak{J}(A) : A \in \Gamma\}$.

Abusing notation slightly, we will often define interpretations simply by defining the truth values of atomic formulas in \mathcal{L}^U .

Definition 2.3. If Γ is a set of formulas (possibly infinite), we say that Γ *entails* A in \mathbf{G}_V , $\Gamma \models_V A$ iff for all \mathfrak{J} into V , $\mathfrak{J}(\Gamma) \leq \mathfrak{J}(A)$.

Γ *1-entails* A in \mathbf{G}_V , $\Gamma \Vdash_V A$, iff, for all \mathfrak{J} into V , whenever $\mathfrak{J}(B) = 1$ for all $B \in \Gamma$, then $\mathfrak{J}(A) = 1$.

We will write $\Gamma \models A$ instead of $\Gamma \models_V A$ in case it is obvious which truth value set V is meant.

Definition 2.4. For a Gödel set V we define the *first order Gödel logic* \mathbf{G}_V as the set of all pairs (Γ, A) such that $\Gamma \models_V A$.

Remark 2.5. We take the notion of *entailment* as the fundamental model-theoretic notion for Gödel logics, in favor of the notion of satisfiability (where Γ is *satisfiable* if there is an interpretation \mathfrak{J} so that $\mathfrak{J}(A) = 1$ for all $A \in \Gamma$). We do this for two reasons:

First, the notion of satisfiability by itself is not particularly interesting in the case of Gödel logics based on Gödel sets V where 0 is isolated: in this case, a set of formulas Γ is satisfiable iff it is satisfiable classically. (This follows from Lemma 2.14

proved below.) This is, however, not the case if 0 is not isolated in V , e.g., $\neg\forall xP(x) \wedge \forall x\neg\neg P(x)$ is satisfiable in $\mathbf{G}_{\mathbb{R}}$, but not classically.

Second, in the case of Gödel logics, the connection between satisfiability and entailment one is used to from classical logic breaks down. It is not the case that $\models A$ iff $\{\neg A\}$ is unsatisfiable. For instance, $B \vee \neg B$ is not a tautology, but can also never take the value 0, hence $\mathfrak{I}(\neg(B \vee \neg B)) = 0$ for all \mathfrak{I} , i.e., $\neg(B \vee \neg B)$ is unsatisfiable. So entailment cannot be defined in terms of satisfiability in the same way as in classical logic. Yet, satisfiability can be defined in terms of entailment: Γ is satisfiable iff $\Gamma \not\models \perp$. Hence also for Gödel logics, establishing soundness and strong completeness for entailment yields the familiar versions of soundness and completeness in terms of satisfiability: a set of formulas Γ is satisfiable iff it is consistent.

We will be concerned below with the relationships between Gödel logics, here considered as entailment relations. Note that $\mathbf{G}_V \models A$ iff $(\emptyset, A) \in \mathbf{G}_V$, so in particular, showing that $\mathbf{G}_V \subseteq \mathbf{G}_W$ also shows that every valid formula of \mathbf{G}_V is also valid in \mathbf{G}_W . On the other hand, to show that $\mathbf{G}_V \not\subseteq \mathbf{G}_W$ it suffices to show that for some A , $\mathbf{G}_V \models A$ but $\mathbf{G}_W \not\models A$.

Remark 2.6. Whether or not a formula A evaluates to 1 under an interpretation \mathfrak{I} depends only on the *relative ordering* of the truth values of the atomic formulas (in $\mathcal{L}^{\mathfrak{I}}$), and not directly on the set V or on the specific values of the atomic formulas. If $V \subseteq W$ are both Gödel sets, and \mathfrak{I} is a V -interpretation, then \mathfrak{I} can be seen also as a W -interpretation, and the values $\mathfrak{I}(A)$, computed recursively using (1)–(7), do not depend on whether we view \mathfrak{I} as a V -interpretation or a W -interpretation. Consequently, if $V \subseteq W$, there are more interpretations into W than into V . Hence, if $\Gamma \models_W A$ then also $\Gamma \models_V A$ and $\mathbf{G}_W \subseteq \mathbf{G}_V$.

This can be generalized to embeddings between Gödel sets other than inclusion. First, we make precise which formulas are involved in the computation of the truth-value of a formula A in an interpretation \mathfrak{I} :

Definition 2.7. The only subformula of an atomic formula A in \mathcal{L}^U is A itself. The subformulas of $A \star B$ for $\star \in \{\rightarrow, \wedge, \vee\}$ are the subformulas of A and of B , together with $A \star B$ itself. The subformulas of $\forall xA(x)$ and $\exists xA(x)$ with respect to a universe U are all subformulas of all $A(u)$ for $u \in U$, together with $\forall xA(x)$ (or, $\exists xA(x)$, respectively) itself.

The set of truth-values of subformulas of A under a given interpretation \mathfrak{I} is denoted by

$$\text{Val}(\mathfrak{I}, A) = \{\mathfrak{I}(B) : B \text{ subformula of } A \text{ w.r.t. } U^{\mathfrak{I}}\} \cup \{0, 1\}$$

If Γ is a set of formulas, then $\text{Val}(\mathfrak{I}, \Gamma) = \bigcup \{\text{Val}(\mathfrak{I}, A) : A \in \Gamma\}$.

Lemma 2.8. *Let \mathfrak{I} be a V -interpretation, and let $h : \text{Val}(\mathfrak{I}, \Gamma) \rightarrow W$ be a mapping satisfying the following properties:*

- (1) $h(0) = 0, h(1) = 1$;
- (2) h is strictly monotonic, i.e., if $a < b$, then $h(a) < h(b)$;
- (3) for every $X \subseteq \text{Val}(\mathfrak{J}, \Gamma)$, $h(\inf X) = \inf h(X)$ and $h(\sup X) = \sup h(X)$ (provided $\inf X, \sup X \in \text{Val}(\mathfrak{J}, \Gamma)$).

Then the W -interpretation \mathfrak{J}_h with universe $U^{\mathfrak{J}}$, $f^{\mathfrak{J}_h} = f^{\mathfrak{J}}$, and for atomic $B \in \mathcal{L}^{\mathfrak{J}}$,

$$\mathfrak{J}_h(B) = \begin{cases} h(\mathfrak{J}(B)) & \text{if } \mathfrak{J}(B) \in \text{dom } h \\ 1 & \text{otherwise} \end{cases}$$

satisfies $\mathfrak{J}_h(A) = h(\mathfrak{J}(A))$ for all $A \in \Gamma$.

PROOF. By induction on the complexity of A . If $A \equiv \perp$, the claim follows from (1). If A is atomic, it follows from the definition of \mathfrak{J}_h . For the propositional connectives the claim follows from the strict monotonicity of h (2). For the quantifiers, it follows from property (3). \square

Remark 2.9. Note that the construction of \mathfrak{J}_h and the proof of Lemma 2.8 also goes through without the condition $h(0) = 0$, provided that the formulas in Γ do not contain \perp , and goes through without the requirement that existing \inf 's be preserved ($h(\inf X) = \inf h(X)$ if $\inf X \in \text{Val}(\mathfrak{J}, \Gamma)$) provided they do not contain \forall .

Definition 2.10. A \mathbf{G} -embedding $h: V \rightarrow W$ is a strictly monotonic, continuous mapping between Gödel sets which preserves 0 and 1.

Lemma 2.11. Suppose $h: V \rightarrow W$ is a \mathbf{G} -embedding. (a) If \mathfrak{J} is a V -interpretation, and \mathfrak{J}_h is the interpretation induced by \mathfrak{J} and h , then $\mathfrak{J}_h(A) = h(\mathfrak{J}(A))$. (b) If $\Gamma \models_W A$ then $\Gamma \models_V A$ (and hence $\mathbf{G}_W \subseteq \mathbf{G}_V$). (c) If h is bijective, then $\Gamma \models_W A$ iff $\Gamma \models_V A$ (and hence, $\mathbf{G}_V = \mathbf{G}_W$).

PROOF. (a) h satisfies the conditions of Lemma 2.8, for Γ the set of all formulas. (b) If $\Gamma \not\models_V A$, then for some \mathfrak{J} , $\mathfrak{J}(B) = 1$ for all $B \in \Gamma$ and $\mathfrak{J}(A) < 1$. By Lemma 2.8, $\mathfrak{J}_h(B) = 1$ for all $B \in \Gamma$ and $\mathfrak{J}_h(A) < 1$ (by strict monotonicity of h). Thus $\Gamma \not\models_W A$. (c) If h is bijective then h^{-1} is also a \mathbf{G} -embedding. \square

Definition 2.12 (Submodel, elementary submodel). Let $\mathfrak{J}_1, \mathfrak{J}_2$ be interpretations. We write $\mathfrak{J}_1 \subseteq \mathfrak{J}_2$ (\mathfrak{J}_2 extends \mathfrak{J}_1) iff $U^{\mathfrak{J}_1} \subseteq U^{\mathfrak{J}_2}$, and for all k , all k -ary predicate symbols P in \mathcal{L} , and all k -ary function symbols f in \mathcal{L} we have

$$P^{\mathfrak{J}_1} = P^{\mathfrak{J}_2} \upharpoonright (U^{\mathfrak{J}_1})^k \quad f^{\mathfrak{J}_1} = f^{\mathfrak{J}_2} \upharpoonright (U^{\mathfrak{J}_1})^k$$

or in other words, if \mathfrak{J}_1 and \mathfrak{J}_2 agree on closed atomic formulas.

We write $\mathfrak{J}_1 \prec \mathfrak{J}_2$ if $\mathfrak{J}_1 \subseteq \mathfrak{J}_2$ and $\mathfrak{J}_1(A) = \mathfrak{J}_2(A)$ for all $\mathcal{L}^{U^{\mathfrak{J}_1}}$ -formulas A .

Proposition 2.13 (Downward Löwenheim-Skolem). *For any interpretation \mathfrak{J} with $U^{\mathfrak{J}}$ infinite, there is an interpretation $\mathfrak{J}' \prec \mathfrak{J}$ with a countable universe $U^{\mathfrak{J}'}$.*

PROOF (sketch). The proof is an easy generalization of the construction for the classical case. We construct a sequence of countable subsets $U_1 \subseteq U_2 \subseteq \dots$ of $U^{\mathfrak{J}}$: U_1 simply contains $t^{\mathfrak{J}}$ for all closed terms of the original language. U_{i+1} is constructed from U_i by adding, for each of the (countably many) formulas of the form $\exists xA(x)$ and $\forall xA(x)$ in the language \mathcal{L}^{U_i} , a countable sequence a_j of elements of $U^{\mathfrak{J}}$ so that $(\mathfrak{J}(A(a_j)))_j \rightarrow \mathfrak{J}(\exists xA(x))$ or $\rightarrow \mathfrak{J}(\forall xA(x))$, respectively. $U^{\mathfrak{J}'} = \bigcup_i U_i$. \square

Lemma 2.14. *Let \mathfrak{J} be an interpretation into V , $w \in [0, 1]$, and let \mathfrak{J}_w be defined by*

$$\mathfrak{J}_w(B) = \begin{cases} \mathfrak{J}(B) & \text{if } \mathfrak{J}(B) < w \\ 1 & \text{otherwise} \end{cases}$$

for atomic formulas B in $\mathcal{L}^{\mathfrak{J}}$. Then \mathfrak{J}_w is an interpretation into V . If $w \notin \text{Val}(\mathfrak{J}, A)$, then $\mathfrak{J}_w(A) = \mathfrak{J}(A)$ if $\mathfrak{J}(A) < w$, and $\mathfrak{J}_w(A) = 1$ otherwise.

PROOF. By induction on the complexity of formulas A in $\mathcal{L}^{\mathfrak{J}}$. The condition that $w \notin \text{Val}(\mathfrak{J}, A)$ is needed to prove the case of $A \equiv \exists xB(x)$, since if $\mathfrak{J}(\exists xB(x)) = w$ and $\mathfrak{J}(B(d)) < w$ for all d , we would have $\mathfrak{J}_w(\exists xB(x)) = w$ and not $= 1$. \square

Proposition 2.15. *Entailment and 1-entailment coincide, i.e., $\Gamma \models A$ iff $\Gamma \Vdash A$*

PROOF. Only if: obvious. If: Suppose that $\Gamma \not\models A$, i.e., there is a V -interpretation \mathfrak{J} so that $\mathfrak{J}(\Gamma) > \mathfrak{J}(A)$. By Proposition 2.13, we may assume that $U^{\mathfrak{J}}$ is countable. Hence, there is some w with $\mathfrak{J}(A) < w < \mathfrak{J}(\Gamma)$ and $w \notin \text{Val}(\mathfrak{J}, \Gamma \cup \{A\})$. Let \mathfrak{J}_w be as in Lemma 2.14. Then $\mathfrak{J}_w(B) = 1$ for all $B \in \Gamma$ and $\mathfrak{J}_w(A) < 1$. \square

The coincidence of the two entailment relations is a unique feature of Gödel logics. Proposition 2.15 does not hold in Łukasiewicz logic, for instance. There, $A, A \rightarrow_{\mathbb{L}} B \Vdash B$ but $A, A \rightarrow_{\mathbb{L}} B \not\models B$. In what follows, we will use \models when semantic entailment is at issue; the preceding proposition shows that the results we obtain for \models hold for \Vdash as well.

Lemma 2.16 (Semantic deduction theorem).

$$\Gamma, A \models B \quad \text{iff} \quad \Gamma \models A \rightarrow B.$$

PROOF. Immediate consequence of the definition of \models and the semantics for \rightarrow . \square

We want to conclude this part with two interesting observations:

Relation to residuated algebras. If one considers the truth value set as a Heyting algebra with $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$, and

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$$

then \rightarrow and \wedge are residuated, i.e.,

$$(a \rightarrow b) = \sup\{x : (x \wedge a) \leq b\}.$$

The Gödel conditional. A large class of many-valued logics can be developed from the theory of t -norms (Hájek, 1998). The class of t -norm based logics includes not only (standard) Gödel logic, but also Łukasiewicz and product logic. In these logics, the conditional is defined as the residuum of the respective t -norm, and the logics differ only in the definition of their t -norm and the respective residuum, i.e., the conditional. The truth function for the Gödel conditional is of particular interest as it can be ‘deduced’ from simple properties of the evaluation and the entailment relation, a fact which was first observed by G. Takeuti.

Lemma 2.17. *Suppose we have a standard language containing a ‘conditional’ \rightarrow interpreted by a truth-function into $[0, 1]$. Suppose further that*

- (1) *a conditional evaluates to 1 if the truth value of the antecedent is less or equal to the truth value of the consequent, i.e., if $\mathfrak{I}(A) \leq \mathfrak{I}(B)$, then $\mathfrak{I}(A \rightarrow B) = 1$;*
- (2) *\vDash is defined as above, i.e., if $\Gamma \vDash B$, then $\mathfrak{I}(\Gamma) \leq \mathfrak{I}(B)$;*
- (3) *the deduction theorem holds, i.e., $\Gamma \cup \{A\} \vDash B \Leftrightarrow \Gamma \vDash A \rightarrow B$.*

Then \rightarrow is the Gödel conditional.

PROOF. From (1), we have that $\mathfrak{I}(A \rightarrow B) = 1$ if $\mathfrak{I}(A) \leq \mathfrak{I}(B)$. Since \vDash is reflexive, $B \vDash B$. Since it is monotonic, $B, A \vDash B$. By the deduction theorem, $B \vDash A \rightarrow B$. By (2),

$$\mathfrak{I}(B) \leq \mathfrak{I}(A \rightarrow B).$$

From $A \rightarrow B \vDash A \rightarrow B$ and the deduction theorem, we get $A \rightarrow B, A \vDash B$. By (2),

$$\min\{\mathfrak{I}(A \rightarrow B), \mathfrak{I}(A)\} \leq \mathfrak{I}(B).$$

Thus, if $\mathfrak{I}(A) > \mathfrak{I}(B)$, $\mathfrak{I}(A \rightarrow B) \leq \mathfrak{I}(B)$. □

Note that all usual conditionals (Gödel, Łukasiewicz, product conditionals) satisfy condition (1). So, in some sense, the Gödel conditional is the only many-valued conditional which validates both directions of the deduction theorem for \vDash . For instance, for the Łukasiewicz conditional $\rightarrow_{\mathbb{L}}$ the right-to-left direction fails: $A \rightarrow_{\mathbb{L}} B \vDash A \rightarrow_{\mathbb{L}} B$, but $A \rightarrow_{\mathbb{L}} B, A \not\vDash B$. (With respect to \Vdash , the left-to-right direction of the deduction theorem fails for $\rightarrow_{\mathbb{L}}$.)

2.2 Axioms and deduction systems

In this section we introduce axioms and deduction systems for Gödel logics, and we will show completeness of these deduction systems subsequently. We will use a Hilbert style proof system:

Definition 2.18. A formula A is derivable from formulas Γ in a system \mathcal{A} consisting of axioms and rules iff there are formulas $A_0, \dots, A_n = A$ such that for each $0 \leq i \leq n$ either $A_i \in \Gamma$, or A_i is an instance of an axiom in \mathcal{A} , or there are indices $j_1, \dots, j_l < i$ and a rule in \mathcal{A} such that A_{j_1}, \dots, A_{j_l} are the premises and A_i is the conclusion of the rule. In this case we write $\Gamma \vdash_{\mathcal{A}} A$.

We will denote by **IL** the following complete axiom system for intuitionistic logic (taken from van Dalen 1986). Rules are written as $A_1, \dots, A_n \vdash A$.

- | | |
|---|---|
| <p>(1) $A \rightarrow (B \rightarrow A)$
 (3) $A \rightarrow (B \rightarrow (A \wedge B))$
 (5) $A \rightarrow (A \vee B), \quad B \rightarrow (A \vee B)$
 (7) $(A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)$
 (9) $\forall x A(x) \rightarrow A(t)$</p> | <p>(2) $(A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C))$
 (4) $(A \wedge B) \rightarrow A, \quad (A \wedge B) \rightarrow B$
 (6) $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C))$
 (8) $A(t) \rightarrow \exists x A(x)$
 (10) $A \rightarrow (\neg A \rightarrow B)$</p> |
|---|---|
- (MP) $A, A \rightarrow B \vdash B$
(∀I) $B^{(x)} \rightarrow A(x) \vdash B^{(x)} \rightarrow \forall x A(x)$
(∃I) $A(x) \rightarrow B^{(x)} \vdash \exists x A(x) \rightarrow B^{(x)}$

Here, the notation $B^{(x)}$ indicates that x is not free in B . Moreover, in (8) and (9), t must be substitutable for x in $A(x)$.

The following axioms will play an important rôle (QS stands for “quantifier shift,” LIN for “linearity,” ISO₀ for “isolation axiom of 0,” and FIN(n) for “finite with n elements”):

- QS $\forall x (C^{(x)} \vee A(x)) \rightarrow (C^{(x)} \vee \forall x A(x))$
LIN $(A \rightarrow B) \vee (B \rightarrow A)$
ISO₀ $\forall x \neg \neg A(x) \rightarrow \neg \neg \forall x A(x)$
FIN(n) $(A_0 \rightarrow A_1) \vee (A_1 \rightarrow A_2) \vee \dots \vee (A_{n-2} \rightarrow A_{n-1}) \vee (A_{n-1} \rightarrow A_n)$

Definition 2.19. \mathbf{H} denotes the axiom system $\mathbf{IL} + \text{QS} + \text{LIN}$.

\mathbf{H}_n for $n \geq 2$ denotes the axiom system $\mathbf{H} + \text{FIN}(n)$.

\mathbf{H}_0 denotes the axiom system $\mathbf{H} + \text{ISO}_0$.

Theorem 2.20 (Soundness). *Suppose Γ contains only closed formulas, and all axioms of \mathcal{A} are valid in \mathbf{G}_V . Then, if $\Gamma \vdash_{\mathcal{A}} A$ then $\Gamma \models_V A$. In particular, \mathbf{H} is sound for \models_V for any Gödel set V ; \mathbf{H}_n is sound for \models_V if $|V| = n$; and \mathbf{H}_0 is sound for \models_V if 0 is isolated in V .*

PROOF. By induction on the complexity of proofs. It is easily verified that $\mathbf{G}_V \models A$ for all axioms of \mathbf{H} and all Gödel sets V ; that if $|V| = n$, then $\mathbf{G}_V \models \text{FIN}(n)$, and that $\mathbf{G}_V \models \text{ISO}_0$ if 0 is isolated in V . So, if A_i is an axiom of \mathcal{A} , $\Gamma \models_V A_i$. If $A_i \in \Gamma$, then obviously $\Gamma \models_V A_i$. It remains to show that the rules of inference preserve consequence.

Suppose $\Gamma \models_V A$ and $\Gamma \models_V A \rightarrow B$ and consider a V -interpretation \mathfrak{J} . Let $v = \mathfrak{J}(\Gamma)$. If $\mathfrak{J}(A) \leq \mathfrak{J}(B)$, then we have $v \leq \mathfrak{J}(B)$ because $v \leq \mathfrak{J}(A)$. If $\mathfrak{J}(A) > \mathfrak{J}(B)$, then $v \leq \mathfrak{J}(B)$ because $\mathfrak{J}(B) = \mathfrak{J}(A \rightarrow B)$.

Suppose $\Gamma \models_V A(x) \rightarrow B$ and x does not occur free in B , but $\Gamma \not\models_V \exists x A(x) \rightarrow B$. Let \mathfrak{J} be a V -interpretation for which $\mathfrak{J}(\Gamma) > \mathfrak{J}(\exists x A(x) \rightarrow B)$. Then $\mathfrak{J}(\exists x A(x)) > \mathfrak{J}(B)$. Set $w = \sup\{\mathfrak{J}(A(d)) : d \in U^{\mathfrak{J}}\}$, and let \mathfrak{J}^d be the interpretation resulting from \mathfrak{J} by assigning d to x . Since the formulas in Γ are all closed and B does not contain x free, $\mathfrak{J}^d(C) = \mathfrak{J}(C)$ for all $C \in \Gamma \cup \{B\}$ and $d \in U^{\mathfrak{J}}$. Now since $w = \mathfrak{J}(\exists x A(x)) > \mathfrak{J}(B)$, there is some $v \in V$, $\mathfrak{J}(B) < v \leq w$ so that $v = \mathfrak{J}(A(d))$ for some $d \in U^{\mathfrak{J}}$. But then $\mathfrak{J}^d(A(x) \rightarrow B) = \mathfrak{J}(B) > \mathfrak{J}^d(\Gamma) = \mathfrak{J}(\Gamma)$, contradicting $\Gamma \models_V A(x) \rightarrow B$. The case for $(\forall I)$ is analogous. \square

Note that the restriction to closed formulas in Γ is essential: $A(x) \vdash_{\mathbf{H}} \forall x A(x)$ but obviously $A(x) \not\models_V \forall x A(x)$.

2.3 Relationships between Gödel logics

The relationships between finite and infinite valued *propositional* Gödel logics are well understood. Any choice of an infinite set of truth-values results in the same set of tautologies, viz., Dummett's \mathbf{LC} . \mathbf{LC} was originally defined using the set of truth-values V_{\downarrow} (see below). Furthermore, we know that \mathbf{LC} is the intersection of all finite-valued propositional Gödel logics, and that it is axiomatized by intuitionistic propositional logic \mathbf{IPL} plus the schema $(A \rightarrow B) \vee (B \rightarrow A)$. \mathbf{IPL} is contained in all Gödel logics.

In the first-order case, the relationships are somewhat more interesting. First of all, let us note the following fact corresponding to the end of the previous paragraph:

Proposition 2.21. *Intuitionistic predicate logic \mathbf{IL} is contained in all first-order Gödel logics.*

PROOF. The axioms and rules of \mathbf{IL} are sound for the Gödel truth functions. \square

As a consequence of this proposition, we will be able to use any intuitionistically sound rule and intuitionistically valid formula when working in any of the Gödel logics.

We now establish some results regarding the relationships between various first-order Gödel logics. For this, it is useful to consider several “prototypical” Gödel sets.

$$\begin{aligned} V_{\mathbb{R}} &= [0, 1] & V_0 &= \{0\} \cup [1/2, 1] \\ V_{\downarrow} &= \{1/k : k \geq 1\} \cup \{0\} \\ V_{\uparrow} &= \{1 - 1/k : k \geq 1\} \cup \{1\} \\ V_n &= \{1 - 1/k : 1 \leq k \leq n-1\} \cup \{1\} \end{aligned}$$

The corresponding Gödel logics are $\mathbf{G}_{\mathbb{R}}$, \mathbf{G}_0 , \mathbf{G}_{\downarrow} , \mathbf{G}_{\uparrow} , and \mathbf{G}_n . $\mathbf{G}_{\mathbb{R}}$ is the *standard* Gödel logic.

The logic \mathbf{G}_{\downarrow} also turns out to be closely related to some temporal logics (Baaz et al., 1996b,a). \mathbf{G}_{\uparrow} is the intersection of all finite-valued first-order Gödel logics as shown in Theorem 2.24.

Proposition 2.22. $\mathbf{G}_{\mathbb{R}} = \bigcap_V \mathbf{G}_V$, where V ranges over all Gödel sets.

PROOF. If $\Gamma \models_V A$ for every Gödel set V , then it does so in particular for $V = [0, 1]$. Conversely, if $\Gamma \not\models_V A$ for a Gödel set V , there is a V -interpretation \mathfrak{I} with $\mathfrak{I}(\Gamma) > \mathfrak{I}(A)$. Since \mathfrak{I} is also a $[0, 1]$ -interpretation, $\Gamma \not\models_{\mathbb{R}} A$. \square

Proposition 2.23. *The following strict containment relationships hold:*

- (1) $\mathbf{G}_n \supsetneq \mathbf{G}_{n+1}$,
- (2) $\mathbf{G}_n \supsetneq \mathbf{G}_{\uparrow} \supsetneq \mathbf{G}_{\mathbb{R}}$,
- (3) $\mathbf{G}_n \supsetneq \mathbf{G}_{\downarrow} \supsetneq \mathbf{G}_{\mathbb{R}}$,
- (4) $\mathbf{G}_0 \supsetneq \mathbf{G}_{\mathbb{R}}$.

PROOF. The only non-trivial part is proving that the containments are strict. For this note that

$$\text{FIN}(n) \equiv (A_0 \rightarrow A_1) \vee \dots \vee (A_{n-1} \rightarrow A_n)$$

is valid in \mathbf{G}_n but not in \mathbf{G}_{n+1} . Furthermore, let

$$\begin{aligned} C_\uparrow &= \exists x(A(x) \rightarrow \forall yA(y)) \text{ and} \\ C_\downarrow &= \exists x(\exists yA(y) \rightarrow A(x)). \end{aligned}$$

C_\downarrow is valid in all \mathbf{G}_n and in \mathbf{G}_\uparrow and \mathbf{G}_\downarrow ; C_\uparrow is valid in all \mathbf{G}_n and in \mathbf{G}_\uparrow , but not in \mathbf{G}_\downarrow ; neither is valid in \mathbf{G}_0 or $\mathbf{G}_\mathbb{R}$ (Baaz et al. 1996b, Corollary 2.9).

$\mathbf{G}_0 \models \text{ISO}_0$ but $\mathbf{G}_\mathbb{R} \not\models \text{ISO}_0$. □

The formulas C_\uparrow and C_\downarrow are of some importance in the study of first-order infinite-valued Gödel logics. C_\uparrow expresses the fact that the infimum of any subset of the set of truth values is contained in the subset (every infimum is a minimum), and C_\downarrow states that every supremum (except possibly 1) is a maximum. The intuitionistically admissible quantifier shifting rules are given by the following implications and equivalences:

$$(\forall xA(x) \wedge B) \leftrightarrow \forall x(A(x) \wedge B) \quad (8)$$

$$(\exists xA(x) \wedge B) \leftrightarrow \exists x(A(x) \wedge B) \quad (9)$$

$$(\forall xA(x) \vee B) \rightarrow \forall x(A(x) \vee B) \quad (10)$$

$$(\exists xA(x) \vee B) \leftrightarrow \exists x(A(x) \vee B) \quad (11)$$

$$(B \rightarrow \forall xA(x)) \leftrightarrow \forall x(B \rightarrow A(x)) \quad (12)$$

$$(B \rightarrow \exists xA(x)) \leftarrow \exists x(B \rightarrow A(x)) \quad (13)$$

$$(\forall xA(x) \rightarrow B) \leftarrow \exists x(A(x) \rightarrow B) \quad (14)$$

$$(\exists xA(x) \rightarrow B) \leftrightarrow \forall x(A(x) \rightarrow B) \quad (15)$$

The remaining three are:

$$(\forall xA(x) \vee B) \leftarrow \forall x(A(x) \vee B) \quad (S_1)$$

$$(B \rightarrow \exists xA(x)) \rightarrow \exists x(B \rightarrow A(x)) \quad (S_2)$$

$$(\forall xA(x) \rightarrow B) \rightarrow \exists x(A(x) \rightarrow B) \quad (S_3)$$

Of these, S_1 is valid in any Gödel logic. S_2 and S_3 imply and are implied by C_\downarrow and C_\uparrow , respectively (take $\exists yA(y)$ and $\forall yA(y)$, respectively, for B). S_2 and S_3 are, respectively, both valid in \mathbf{G}_\uparrow , invalid and valid in \mathbf{G}_\downarrow , and both invalid in $\mathbf{G}_\mathbb{R}$.

Note that since we defined $\neg A \equiv A \rightarrow \perp$, the quantifier shifts for \rightarrow (14, 15, S_3) include the various directions of De Morgan's laws as special cases. Specifically, the only direction of De Morgan's laws which is not valid in all Gödel logics is the one corresponding to (S_3), i.e., $\neg \forall xA(x) \rightarrow \exists x\neg A(x)$. This formula is equivalent to

ISO₀. For, $\mathbf{G}_V \models \forall x \neg \neg A(x) \leftrightarrow \neg \exists x \neg A(x)$ by (15). We get ISO₀ using $\neg \exists x \neg A(x) \rightarrow \neg \neg \forall x A(x)$, which is an instance of (S₃). The other direction is given in Lemma 5.6.

We now also know that $\mathbf{G}_\uparrow \neq \mathbf{G}_\downarrow$. In fact, we have $\mathbf{G}_\downarrow \subsetneq \mathbf{G}_\uparrow$; this follows from the following theorem.

Theorem 2.24.

$$\mathbf{G}_\uparrow = \bigcap_{n \geq 2} \mathbf{G}_n$$

PROOF. By Proposition 2.23, $\mathbf{G}_\uparrow \subseteq \bigcap_{n \geq 2} \mathbf{G}_n$. We now prove the reverse inclusion. Suppose $\Gamma \not\models_{V_\uparrow} A$, i.e., there is a V_\uparrow -interpretation \mathfrak{I} such that $\mathfrak{I}(\Gamma) > \mathfrak{I}(A)$. Let $\mathfrak{I}(A) = 1 - 1/k$, and pick w somewhere between $1 - 1/k$ and $1 - 1/(k+1)$. Then the interpretation \mathfrak{I}_w given by Lemma 2.14 is so that $\mathfrak{I}(\Gamma) = 1$ and $\mathfrak{I}(A) = 1 - 1/k$. Since there are only finitely many truth values below w in V_\uparrow , \mathfrak{I}_w is also a \mathbf{G}_{k+1} interpretation which shows that $\Gamma \not\models_{V_{k+1}} A$. Hence, $(\Gamma, A) \notin \bigcap_{n \geq 2} \mathbf{G}_n$. \square

Corollary 2.25. $\mathbf{G}_n \supseteq \bigcap_n \mathbf{G}_n = \mathbf{G}_\uparrow \supseteq \mathbf{G}_\downarrow \supseteq \mathbf{G}_\mathbb{R}$

Note that also $\mathbf{G}_\uparrow \supseteq \mathbf{G}_0 \supseteq \mathbf{G}_\mathbb{R}$ by the above, and that neither $\mathbf{G}_0 \subseteq \mathbf{G}_\downarrow$ nor $\mathbf{G}_\downarrow \subseteq \mathbf{G}_0$ (counterexamples are ISO₀ or $\neg \forall x A(x) \rightarrow \exists \neg A(x)$, and C_\downarrow , respectively).

As we will see later, the axioms FIN(n) axiomatize exactly the finite-valued Gödel logics. In these logics the quantifier shift axiom QS is not necessary. Furthermore, all quantifier shift rules are valid in the finite valued logics. Since \mathbf{G}_\uparrow is the intersection of all the finite ones, all quantifier shift rules are valid in \mathbf{G}_\uparrow . Moreover, any infinite-valued Gödel logic other than \mathbf{G}_\uparrow is defined by some V which either contains an infimum which is not a minimum, or a supremum (other than 1) which is not a maximum. Hence, in V either C_\uparrow or C_\downarrow will be invalid, and therewith either S_3 or S_2 . We have:

Corollary 2.26. *In \mathbf{G}_V all quantifier shift rules are valid iff there is a \mathbf{G} -embedding from V to V_\uparrow , i.e., V is either finite or order isomorphic to V_\uparrow .*

This means that it is in general not possible to transform formulas to equivalent prenex formulas in the usual way. Moreover, in general there is not even a recursive procedure for mapping formulas to equivalent, or even just validity-equivalent formulas in prenex form, since for some V , \mathbf{G}_V is not r.e. whereas the corresponding prenex fragment is r.e. (see Section 7.1).

3 Topology and Order

3.1 Perfect sets

All the following notations, lemmas, theorems are carried out within the framework of Polish spaces, which are separable, completely metrizable topological spaces. For our discussion it is only necessary to know that \mathbb{R} and all its closed subsets are Polish spaces (hence, every Gödel set is a Polish space). For a detailed exposition see Moschovakis (1980) and Kechris (1995).

Definition 3.1 (Limit point, perfect space, perfect set). A *limit point* of a topological space is a point that is not isolated, i.e. for every open neighborhood U of x there is a point $y \in U$ with $y \neq x$. A space is *perfect* if all its points are limit points. A set $P \subseteq \mathbb{R}$ is *perfect* if it is closed and together with the topology induced from \mathbb{R} is a perfect space.

It is obvious that all (non-trivial) closed intervals are perfect sets, as well as all countable unions of (non-trivial) intervals. But all these sets generated from closed intervals have the property that they are “everywhere dense,” i.e., contained in the closure of their inner component. There is a well-known example of a perfect set that is nowhere dense, the Cantor set:

Example 3.2 (Cantor Set). The set of all numbers in the unit interval which can be expressed in triadic notation only by digits 0 and 2 is called the *Cantor set* \mathbb{D} .

A more intuitive way to obtain this set is to start with the unit interval, take out the open middle third and restart this process with the lower and the upper third. Repeating this you get exactly the Cantor set because the middle third always contains the numbers which contain the digit 1 in their triadic notation.

This set has a lot of interesting properties, the most important one for our purposes is that it is a perfect set:

Proposition 3.3. *The Cantor set is perfect.*

It is possible to embed the Cauchy space into any perfect space, yielding the following proposition:

Proposition 3.4 (Kechris 1995, Corollary 6.3). *If X is a nonempty perfect Polish space, then $|X| = 2^{\aleph_0}$. All nonempty perfect subsets of $[0, 1]$ have cardinality 2^{\aleph_0} .*

It is possible to obtain the following characterization of perfect sets (see Winkler 1999):

Proposition 3.5 (Characterization of perfect sets in \mathbb{R}). *For any perfect subset of*

\mathbb{R} there is a unique partition of the real line into countably many intervals such that the intersections of the perfect set with these intervals are either empty, the full interval or isomorphic to the Cantor set.

So we see that intervals and Cantor sets are prototypical for perfect sets and the basic building blocks of more complex perfect sets.

Every Polish space can be partitioned into a perfect kernel and a countable rest. This is the well known Cantor-Bendixon Theorem:

Theorem 3.6 (Cantor-Bendixon). *Let X be a Polish space. Then X can be uniquely written as $X = P \cup C$, with P a perfect subset of X and C countable and open. The subset P is called the perfect kernel of X (denoted by X^∞).*

As a corollary we obtain that any uncountable Polish space contains a perfect set, and therefore, has cardinality 2^{\aleph_0} .

3.2 Relation to Gödel logics

The following lemma was originally proved in Preining (2003), where it was used to extend the proof of recursive axiomatizability of the ‘standard’ Gödel logic $\mathbf{G}_{\mathbb{R}}$ to Gödel logics with a truth value set containing a perfect set in the general case. The following simpler proof is inspired by Beckmann et al. (200?):

Lemma 3.7. *Suppose that $M \subseteq [0, 1]$ is countable and $P \subseteq [0, 1]$ is perfect. Then there is a strictly monotone continuous map $h: M \rightarrow P$ (i.e., infima and suprema already existing in M are preserved). Furthermore, if $\inf M \in M$, then one can choose h such that $h(\inf M) = \inf P$.*

PROOF. Let σ be the mapping which scales and shifts M into $[0, 1]$, i.e. the mapping $x \mapsto (x - \inf M) / (\sup M - \inf M)$ (assuming that M contains more than one point). Let w be an injective monotone map from $\sigma(M)$ into 2^ω , i.e. $w(m)$ is a fixed binary representation of m . For dyadic rational numbers (i.e. those with different binary representations) we fix one possible.

Let i be the natural bijection from 2^ω (the set of infinite $\{0, 1\}$ -sequences, ordered lexicographically) onto \mathbb{D} , the Cantor set. i is an order preserving homeomorphism. Since P is perfect, we can find a continuous strictly monotone map c from the Cantor set $\mathbb{D} \subseteq [0, 1]$ into P , and c can be chosen so that $c(0) = \inf P$. Now $h = c \circ i \circ w \circ \sigma$ is also a strictly monotone map from M into P , and $h(\inf M) = \inf P$, if $\inf M \in M$. Since c is continuous, existing infima and suprema are preserved. \square

Corollary 3.8. *A Gödel set V is uncountable iff it contains a non-trivial dense linear subordering.*

PROOF. If: Every countable non-trivial dense linear order has order type η , $\mathbf{1} + \eta$, $\eta + \mathbf{1}$, or $\mathbf{1} + \eta + \mathbf{1}$ (Rosenstein, 1982, Corollary 2.9), where η is the order type of \mathbb{Q} . The completion of any ordering of order type η has order type λ , the order type of \mathbb{R} (Rosenstein, 1982, Theorem 2.30), thus the truth value set must be uncountable.

Only if: By Theorem 3.6, V^∞ is non-empty. Take $M = \mathbb{Q} \cap [0, 1]$ and $P = V^\infty$ in Lemma 3.7. The image of M under h is a non-trivial dense linear subordering in V . \square

Theorem 3.9. *Suppose V is a truth value set with non-empty perfect kernel P , and let $W = V \cup [\inf P, 1]$. Then $\Gamma \models_V A$ iff $\Gamma \models_W A$, i.e., $\mathbf{G}_V = \mathbf{G}_W$.*

PROOF. As $V \subseteq W$ we have $\mathbf{G}_W \subseteq \mathbf{G}_V$ (cf. Remark 2.6). Now assume that \mathfrak{J} is a W -interpretation which shows that $\Gamma \models_W A$ does *not* hold, i.e., $\mathfrak{J}(\Gamma) > \mathfrak{J}(A)$. By Proposition 2.13, we may assume that $U^\mathfrak{J}$ is countable. The set $\text{Val}(\mathfrak{J}, \Gamma \cup A)$ has cardinality at most \aleph_0 , thus there is a $w \in [0, 1]$ such that $w \notin \text{Val}(\mathfrak{J}, \Gamma \cup A)$ and $\mathfrak{J}(A) < w < 1$. By Lemma 2.14, $\mathfrak{J}_w(A) < w < 1$. Now consider $M = \text{Val}(\mathfrak{J}_w, \Gamma \cup A)$: these are all the truth values from $W = V \cup [\inf P, 1]$ required to compute $\mathfrak{J}_w(A)$ and $\mathfrak{J}_w(B)$ for all $B \in \Gamma$. We have to find some way to map them to V so that the induced interpretation is a counterexample to $\Gamma \models_V A$.

Let $M_0 = M \cap [0, \inf P)$ and $M_1 = (M \cap [\inf P, w]) \cup \{\inf P\}$. By Lemma 3.7 there is a strictly monotone continuous (i.e. preserving all existing infima and suprema) map h from M_1 into P . Furthermore, we can choose h such that $h(\inf M_1) = \inf P$.

We define a function g from $\text{Val}(\mathfrak{J}_w, \Gamma \cup A)$ to V as follows:

$$g(x) = \begin{cases} x & 0 \leq x \leq \inf P \\ h(x) & \inf P \leq x \leq w \\ 1 & x = 1 \end{cases}$$

Note that there is no $x \in \text{Val}(\mathfrak{J}_w, \Gamma \cup A)$ with $w < x < 1$. This function has the following properties: $g(0) = 0$, $g(1) = 1$, g is strictly monotonic and preserves existing infima and suprema. Using Lemma 2.8 we obtain that \mathfrak{J}_g is a V -interpretation with $\mathfrak{J}_g(C) = g(\mathfrak{J}_w(C))$ for all $C \in \Gamma \cup A$, thus also $\mathfrak{J}_g(\Gamma) > \mathfrak{J}_g(A)$. \square

4 Countable Gödel sets

In this section we show that the first-order Gödel logics where the set of truth values does not contain a dense subset are not r.e. We establish this result by reducing the classical validity of a formula in all finite models to the validity of a formula in Gödel logic (the set of these formulas is not r.e. by Trakhtenbrot's Theorem).

Definition 4.1. A formula is called *crisp* if all its atomic subformulas occur either negated or double-negated in it.

Lemma 4.2. *If A and B are crisp and classically equivalent, then also $\mathbf{G}_V \models A \leftrightarrow B$, for any Gödel set V . Specifically, if $A(x)$ and $B^{(x)}$ are crisp, then*

$$\begin{aligned} \mathbf{G}_V \models (\forall x A(x) \rightarrow B^{(x)}) &\leftrightarrow \exists x (A(x) \rightarrow B^{(x)}) \quad \text{and} \\ \mathbf{G}_V \models (B^{(x)} \rightarrow \exists x A(x)) &\leftrightarrow \exists x (B^{(x)} \rightarrow A(x)). \end{aligned}$$

PROOF. Given an interpretation \mathfrak{I} , define $\mathfrak{I}'(C) = 1$ if $\mathfrak{I}(C) > 0$ and $= 0$ if $\mathfrak{I}(C) = 0$ for atomic C . It is easily seen that if A, B are crisp, then $\mathfrak{I}(A) = \mathfrak{I}'(A)$ and $\mathfrak{I}(B) = \mathfrak{I}'(B)$. But \mathfrak{I}' is a classical interpretation, so by assumption $\mathfrak{I}'(A) = \mathfrak{I}'(B)$. \square

Theorem 4.3. *If V is countably infinite, then the set of validities of \mathbf{G}_V is not r.e.*

PROOF. By Theorem 3.8, V is countably infinite iff it is infinite and does not contain a non-trivial densely ordered subset. We show that for every sentence A there is a sentence A^g s.t. A^g is valid in \mathbf{G}_V iff A is true in every finite (classical) first-order structure.

We define A^g as follows: Let P be a unary and L be a binary predicate symbol not occurring in A and let Q_1, \dots, Q_n be all the predicate symbols in A . We use the abbreviations $x \in y \equiv \neg\neg L(x, y)$ and $x \prec y \equiv (P(y) \rightarrow P(x)) \rightarrow P(y)$. Note that for any interpretation \mathfrak{I} , $\mathfrak{I}(x \in y)$ is either 0 or 1, and as long as $\mathfrak{I}(P(x)) < 1$ for all x (in particular, if $\mathfrak{I}(\exists z P(z)) < 1$), we have $\mathfrak{I}(x \prec y) = 1$ iff $\mathfrak{I}(P(x)) < \mathfrak{I}(P(y))$. Let $A^g \equiv$

$$\left\{ \begin{array}{l} S \wedge c_1 \in 0 \wedge c_2 \in 0 \wedge c_2 \prec c_1 \wedge \\ \forall i [\forall x, y \forall j \forall k \exists z D \vee \forall x \neg(x \in s(i))] \end{array} \right\} \rightarrow (A' \vee \exists u P(u))$$

where S is the conjunction of the standard axioms for 0, successor and \leq , with double negations in front of atomic formulas,

$$\begin{aligned} D \equiv & (j \leq i \wedge x \in j \wedge k \leq i \wedge y \in k \wedge x \prec y) \rightarrow \\ & \rightarrow (z \in s(i) \wedge x \prec z \wedge z \prec y) \end{aligned}$$

and A' is A where every atomic formula is replaced by its double negation, and all quantifiers are relativized to the predicate $R(i) \equiv \exists x(x \in i)$.

Intuitively, L is a predicate that divides a subset of the domain into levels, and $x \in i$ means that x is an element of level i . If the antecedent is true, then the true standard axioms S force the domain to be a model of the reduct of PA to the language without $+$ and \times , which could be either a standard model (isomorphic to \mathbb{N}) or a non-standard model (\mathbb{N} followed by copies of \mathbb{Z}). P orders the elements of the domain which fall into one of the levels in a subordering of the truth values.

The idea is that for any two elements in a level $\leq i$ there is an element in a not-empty level $j \geq i$ which lies strictly between those two elements in the ordering given by \prec . If this condition cannot be satisfied, the levels above i are empty. Clearly, this condition can be satisfied in an interpretation \mathfrak{J} only for finitely many levels if V does not contain a dense subset, since if more than finitely many levels are non-empty, then $\bigcup_i \{\mathfrak{J}(P(d)) : \mathfrak{J} \models d \in i\}$ gives a dense subset. By relativizing the quantifiers in A to the indices of non-empty levels, we in effect relativize to a finite subset of the domain. We make this more precise:

Suppose A is classically false in some finite structure \mathfrak{J} . W.l.o.g. we may assume that the domain of this structure is the naturals $0, \dots, n$. We extend \mathfrak{J} to a \mathbf{G}_V -interpretation \mathfrak{J}^g with domain \mathbb{N} as follows: Since V contains infinitely many values, we can choose c_1, c_2, L and P so that $\exists x(x \in i)$ is true for $i = 0, \dots, n$ and false otherwise, and so that $\mathfrak{J}^g(\exists x P(x)) < 1$. The number-theoretic symbols receive their natural interpretation. The antecedent of A^g clearly receives the value 1, and the consequent receives $\mathfrak{J}^g(\exists x P(x)) < 1$, so $\mathfrak{J}^g \not\models A^g$.

Now suppose that $\mathfrak{J} \not\models A^g$. Then $\mathfrak{J}(\exists x P(x)) < 1$. In this case, $\mathfrak{J}(x \prec y) = 1$ iff $\mathfrak{J}(P(x)) < \mathfrak{J}(P(y))$, so \prec defines a strict order on the domain of \mathfrak{J} . It is easily seen that in order for the value of the antecedent of A^g under \mathfrak{J} to be greater than that of the consequent, it must be $= 1$ (the values of all subformulas are either $\leq \mathfrak{J}(\exists x P(x))$ or $= 1$). For this to happen, of course, what the antecedent is intended to express must actually be true in \mathfrak{J} , i.e., that $x \in i$ defines a series of levels and any level $i > 0$ is either empty, or for all x and y occurring in some smaller level there is a z with $x \prec z \prec y$ and $z \in i$.

To see this, consider the relevant part of the antecedent, $B = \forall i [\forall x, y \forall j \forall k \exists z D \vee \forall x \neg(x \in i)]$. If $\mathfrak{J}(B) = 1$, then for all i , either $\mathfrak{J}(\forall x, y \forall j \forall k \exists z D) = 1$ or $\mathfrak{J}(\forall x \neg(x \in i)) = 1$. In the first case, we have $\mathfrak{J}(\exists z D) = 1$ for all x, y, j , and k . Now suppose that for all z , $\mathfrak{J}(D) < 1$, yet $\mathfrak{J}(\exists z D) = 1$. Then for at least some z the value of that formula would have to be $> \mathfrak{J}(\exists z P(z))$, which is impossible. Thus, for every x, y, j, k , there is a z such that $\mathfrak{J}(D) = 1$. But this means that for all x, y s.t. $x \in j, y \in k$ with $j, k \leq i$ and $x \prec y$ there is a z with $x \prec z \prec y$ and $z \in i + 1$.

In the second case, where $\mathfrak{J}(\forall x \neg(x \in i)) = 1$, we have that $\mathfrak{J}(\neg(x \in i)) = 1$ for all x , hence $\mathfrak{J}(x \in i) = 0$ and level i is empty.

Note that the non-empty levels can be distributed over the whole range of the non-standard model, but since V contains no dense subset, the total number of non-empty levels is finite. Thus, A is false in the classical interpretation \mathfrak{I}^c obtained from \mathfrak{I} by restricting \mathfrak{I} to the domain $\{i : \exists x(x \in i)\}$ and $\mathfrak{I}^c(Q) = \mathfrak{I}(\neg\neg Q)$ for atomic Q . \square

This shows that no infinite-valued Gödel logic whose set of truth values does not contain a dense subset, i.e., no countably infinite Gödel logic, is r.e. We strengthen this result in Section 7.1 to show that the prenex fragments are likewise not axiomatizable.

5 Uncountable Gödel sets

5.1 0 is contained in the perfect kernel

If V is uncountable, and 0 is contained in V^∞ , then \mathbf{G}_V is axiomatizable. Indeed, Theorem 3.9 showed that all such logics \mathbf{G}_V coincide. Thus, it is only necessary to establish completeness of the axioms system \mathbf{H} with respect to $\mathbf{G}_\mathbb{R}$. This result has been shown by several people over the years. We give here a generalization of the proof of Takano (1987). Alternative proofs can be found in Horn (1969), Takeuti and Titani (1984), and Hájek (1998). The proof of Horn (1969), however, does not give strong completeness, while the proof of Takeuti and Titani (1984) is specific to the Gödel set $[0, 1]$. Our proof is self-contained and applies to Gödel logics directly, making an extension of the result easier.

Theorem 5.1 (Strong completeness of Gödel logic). *If $\Gamma \vDash_{\mathbb{R}} A$, then $\Gamma \vdash_{\mathbf{H}} A$.*

PROOF. Assume that $\Gamma \not\vdash A$, we construct an interpretation \mathfrak{I} in which $\mathfrak{I}(A) = 1$ for all $B \in \Gamma$ and $\mathfrak{I}(A) < 1$. Let y_1, y_2, \dots be a sequence of free variables which do not occur in $\Gamma \cup \Delta$, let \mathcal{T} be the set of all terms in the language of $\Gamma \cup \Delta$ together with the new variables y_1, y_2, \dots , and let $\mathcal{F} = \{F_1, F_2, \dots\}$ be an enumeration of the formulas in this language in which y_i does not appear in F_1, \dots, F_i and in which each formula appears infinitely often.

If Δ is a set of formulas, we write $\Gamma \Rightarrow \Delta$ if for some $A_1, \dots, A_n \in \Gamma$, and some $B_1, \dots, B_m \in \Delta$, $\vdash_{\mathbf{H}} (A_1 \wedge \dots \wedge A_n) \rightarrow (B_1 \vee \dots \vee B_m)$ (and $\not\Rightarrow$ if this is not the case). We define a sequence of sets of formulas Γ_n, Δ_n such that $\Gamma_n \not\Rightarrow \Delta_n$ by induction. First, $\Gamma_0 = \Gamma$ and $\Delta_0 = \{A\}$. By the assumption of the theorem, $\Gamma_0 \not\Rightarrow \Delta_0$.

If $\Gamma_n \Rightarrow \Delta_n \cup \{F_n\}$, then $\Gamma_{n+1} = \Gamma_n \cup \{F_n\}$ and $\Delta_{n+1} = \Delta_n$. In this case, $\Gamma_{n+1} \not\Rightarrow \Delta_{n+1}$, since otherwise we would have $\Gamma_n \Rightarrow \Delta_n \cup \{F_n\}$ and $\Gamma_n \cup \{F_n\} \Rightarrow \Delta_n$. But

then, we'd have that $\Gamma_n \Rightarrow \Delta_n$, which contradicts the induction hypothesis (note that $\vdash_{\mathbf{H}} (A \rightarrow B \vee F) \rightarrow ((A \wedge F \rightarrow B) \rightarrow (A \rightarrow B))$).

If $\Gamma_n \not\Rightarrow \Delta_n \cup \{F_n\}$, then $\Gamma_{n+1} = \Gamma_n$ and $\Delta_{n+1} = \Delta_n \cup \{F_n, B(y_n)\}$ if $F_n \equiv \forall x B(x)$, and $\Delta_{n+1} = \Delta_n \cup \{F_n\}$ otherwise. In the latter case, it is obvious that $\Gamma_{n+1} \not\Rightarrow \Delta_{n+1}$. In the former, observe that by I10 and QS, if $\Gamma_n \Rightarrow \Delta_n \cup \{\forall x B(x), B(y_n)\}$ then also $\Gamma_n \Rightarrow \Delta_n \cup \{\forall x B(x)\}$ (note that y_n does not occur in Γ_n or Δ_n).

Let $\Gamma^* = \bigcup_{i=0}^{\infty} \Gamma_i$ and $\Delta^* = \bigcup_{i=0}^{\infty} \Delta_i$. We have:

- (1) $\Gamma^* \not\Rightarrow \Delta^*$, for otherwise there would be a k so that $\Gamma_k \Rightarrow \Delta_k$.
- (2) $\Gamma \subseteq \Gamma^*$ and $\Delta \subseteq \Delta^*$ (by construction).
- (3) $\Gamma^* = \mathcal{F} \setminus \Delta^*$, since each F_n is either in Γ_{n+1} or Δ_{n+1} , and if for some n , $F_n \in \Gamma^* \cap \Delta^*$, there would be a k so that $F_n \in \Gamma_k \cap \Delta_k$, which is impossible since $\Gamma_k \not\Rightarrow \Delta_k$.
- (4) If $\Gamma^* \Rightarrow B_1 \vee \dots \vee B_n$, then $B_i \in \Gamma^*$ for some i . For suppose not, then for $i = 1, \dots, n$, $B_i \notin \Gamma^*$, and hence, by (3), $B_i \in \Delta^*$. But then $\Gamma^* \Rightarrow \Delta^*$, contradicting (1).
- (5) If $B(t) \in \Gamma^*$ for every $t \in \mathcal{T}$, then $\forall x B(x) \in \Gamma^*$. Otherwise, by (3), $\forall x B(x) \in \Delta^*$ and so there is some n so that $\forall x B(x) = F_n$ and Δ_{n+1} contains $\forall x B(x)$ and $B(y_n)$. But, again by (3), then $B(y_n) \notin \Gamma^*$.
- (6) Γ^* is closed under provable implication, since if $\Gamma^* \Rightarrow A$, then $A \notin \Delta^*$ and so, again by (3), $A \in \Gamma^*$. In particular, if $\vdash_{\mathbf{H}} A$, then $A \in \Gamma^*$.

Define relations \preceq and \equiv on \mathcal{F} by

$$B \preceq C \Leftrightarrow B \rightarrow C \in \Gamma^* \quad \text{and} \quad B \equiv C \Leftrightarrow B \preceq C \wedge C \preceq B.$$

Then \preceq is reflexive and transitive, since for every B , $\vdash_{\mathbf{H}} B \rightarrow B$ and so $B \rightarrow B \in \Gamma^*$, and if $B \rightarrow C \in \Gamma^*$ and $C \rightarrow D \in \Gamma^*$ then $B \rightarrow D \in \Gamma^*$, since $B \rightarrow C, C \rightarrow D \Rightarrow B \rightarrow D$ (recall (6) above). Hence, \equiv is an equivalence relation on \mathcal{F} . For every B in \mathcal{F} we let $|B|$ be the equivalence class under \equiv to which B belongs, and \mathcal{F}/\equiv the set of all equivalence classes. Next we define the relation \leq on \mathcal{F}/\equiv by

$$|B| \leq |C| \Leftrightarrow B \preceq C \Leftrightarrow B \rightarrow C \in \Gamma^*.$$

Obviously, \leq is independent of the choice of representatives A, B .

Lemma 5.2. $\langle \mathcal{F}/\equiv, \leq \rangle$ is a countably linearly ordered structure with distinct maximal element $|\top|$ and minimal element $|\perp|$.

PROOF. Since \mathcal{F} is countably infinite, \mathcal{F}/\equiv is countable. For every B and C , $\vdash_{\mathbf{H}} (B \rightarrow C) \vee (C \rightarrow B)$ by LIN, and so either $B \rightarrow C \in \Gamma^*$ or $C \rightarrow B \in \Gamma^*$ (by (4)), hence \leq is linear. For every B , $\vdash_{\mathbf{H}} B \rightarrow \top$ and $\vdash_{\mathbf{H}} \perp \rightarrow B$, and so $B \rightarrow \top \in \Gamma^*$ and $\perp \rightarrow B \in \Gamma^*$, hence $|\top|$ and $|\perp|$ are the maximal and minimal elements, respectively. Pick any A in Δ^* . Since $\top \rightarrow \perp \Rightarrow A$, and $A \notin \Gamma^*$, $\top \rightarrow \perp \notin \Gamma^*$, so $|\top| \not\leq |\perp|$. \square

We abbreviate $|\top|$ by $\mathbf{1}$ and $|\perp|$ by $\mathbf{0}$.

Lemma 5.3. *The following properties hold in $\langle \mathcal{F} / \equiv, \leq \rangle$:*

- (1) $|B| = \mathbf{1} \Leftrightarrow B \in \Gamma^*$.
- (2) $|B \wedge C| = \min\{|B|, |C|\}$.
- (3) $|B \vee C| = \max\{|B|, |C|\}$.
- (4) $|B \rightarrow C| = \mathbf{1}$ if $|B| \leq |C|$, $|B \rightarrow C| = |C|$ otherwise.
- (5) $|\neg B| = \mathbf{1}$ if $|B| = \mathbf{0}$; $|\neg B| = \mathbf{0}$ otherwise.
- (6) $|\exists x B(x)| = \sup\{|B(t)| : t \in \mathcal{T}\}$.
- (7) $|\forall x B(x)| = \inf\{|B(t)| : t \in \mathcal{T}\}$.

PROOF. (1) If $|B| = \mathbf{1}$, then $\top \rightarrow B \in \Gamma^*$, and hence $B \in \Gamma^*$. And if $B \in \Gamma^*$, then $\top \rightarrow B \in \Gamma^*$ since $B \Rightarrow \top \rightarrow B$. So $|\top| \leq |B|$. It follows that $|\top| = |B|$ as also $|B| \leq |\top|$.

(2) From $\Rightarrow B \wedge C \rightarrow B, \Rightarrow B \wedge C \rightarrow C$ and $D \rightarrow B, D \rightarrow C \Rightarrow D \rightarrow B \wedge C$ for every D , it follows that $|B \wedge C| = \inf\{|B|, |C|\}$, from which (2) follows since \leq is linear. (3) is proved analogously.

(4) If $|B| \leq |C|$, then $B \rightarrow C \in \Gamma^*$, and since $\top \in \Gamma^*$ as well, $|B \rightarrow C| = \mathbf{1}$. Now suppose that $|B| \not\leq |C|$. From $B \wedge (B \rightarrow C) \Rightarrow C$ it follows that $\min\{|B|, |B \rightarrow C|\} \leq |C|$. Because $|B| \not\leq |C|$, $\min\{|B|, |B \rightarrow C|\} \neq |B|$, hence $|B \rightarrow C| \leq |C|$. On the other hand, $\vdash C \rightarrow (B \rightarrow C)$, so $|C| \leq |B \rightarrow C|$.

(5) If $|B| = \mathbf{0}$, $\neg B = B \rightarrow \perp \in \Gamma^*$, and hence $|\neg B| = \mathbf{1}$ by (1). Otherwise, $|B| \not\leq |\perp|$, and so by (4), $|\neg B| = |B \rightarrow \perp| = \mathbf{0}$.

(6) Since $\vdash_{\mathbf{H}} B(t) \rightarrow \exists x B(x)$, $|B(t)| \leq |\exists x B(x)|$ for every $t \in \mathcal{T}$. On the other hand, for every D without x free,

$$\begin{array}{ll}
|B(t)| \leq |D| & \text{for every } t \in \mathcal{T} \\
\Leftrightarrow B(t) \rightarrow D \in \Gamma^* & \text{for every } t \in \mathcal{T} \\
\Rightarrow \forall x (B(x) \rightarrow D) \in \Gamma^* & \text{by property (5) of } \Gamma^* \\
\Rightarrow \exists x B(x) \rightarrow D \in \Gamma^* & \text{since } \forall x (B(x) \rightarrow D) \Rightarrow \exists x B(x) \rightarrow D \\
\Leftrightarrow |\exists x B(x)| \leq |D|. &
\end{array}$$

(7) is proved analogously. □

$\langle \mathcal{F} / \equiv, \leq \rangle$ is countable, let $\mathbf{0} = a_0, \mathbf{1} = a_1, a_2, \dots$ be an enumeration. Define $h(\mathbf{0}) = 0, h(\mathbf{1}) = 1$, and define $h(a_n)$ inductively for $n > 1$: Let $a_n^- = \max\{a_i : i < n \text{ and } a_i < a_n\}$ and $a_n^+ = \min\{a_i : i < n \text{ and } a_i > a_n\}$, and define $h(a_n) = (h(a_n^-) + h(a_n^+))/2$ (thus, $a_2^- = \mathbf{0}$ and $a_2^+ = \mathbf{1}$ as $\mathbf{0} = a_0 < a_2 < a_1 = \mathbf{1}$, hence $h(a_2) = \frac{1}{2}$). Then $h: \langle \mathcal{F} / \equiv, \leq \rangle \rightarrow \mathbb{Q} \cap [0, 1]$ is a strictly monotone map which preserves infs and

sup. By Lemma 3.7 there exists a \mathbf{G} -embedding h' from $\mathbb{Q} \cap [0, 1]$ into $[0, 1]$ which is also strictly monotone and preserves infs and sups. Put $\mathfrak{J}(B) = h'(h(|B|))$ for every atomic $B \in \mathcal{F}$ and we obtain a $V_{\mathbb{R}}$ -interpretation.

Note that for every B , $\mathfrak{J}(B) = 1$ iff $|B| = \mathbf{1}$ iff $B \in \Gamma^*$. Hence, we have $\mathfrak{J}(B) = 1$ for all $B \in \Gamma$ while if $A \notin \Gamma^*$, then $\mathfrak{J}(A) < 1$, so $\Gamma \not\models A$. Thus we have proven that on the assumption that if $\Gamma \not\models A$, then $\Gamma \not\models A$ \square

This completeness proof can be adapted to hypersequent calculi for Gödel logics (Baaz et al., 2003a; Ciabattoni, 2005), even including the Δ projection operator (Baaz et al., 2006).

As already mentioned we obtain from this completeness proof together with the soundness theorem (Theorem 2.20) and Theorem 3.9 the characterization of recursive axiomatizability:

Theorem 5.4. *Let V be a Gödel set with 0 contained in the perfect kernel of V . Suppose that Γ is a set of closed formulas. Then $\Gamma \models_V A$ iff $\Gamma \vdash_{\mathbf{H}} A$.*

Corollary 5.5 (Deduction theorem for Gödel logics). *Suppose that Γ is a set of formulas, and A is a closed formula. Then*

$$\Gamma, A \vdash_{\mathbf{H}} B \quad \text{iff} \quad \Gamma \vdash_{\mathbf{H}} A \rightarrow B.$$

PROOF. Use the soundness theorem (Theorem 2.20), completeness theorem (Theorem 5.4) and the semantic deduction theorem 2.16. Another proof would be by induction on the length of the proof. See Hájek (1998), Theorem 2.2.18. \square

5.2 0 is isolated

In the case where 0 is isolated in V , and thus also not contained in the perfect kernel, we will transform a counter example in $\mathbf{G}_{\mathbb{R}}$ for $\Gamma, \Pi \models A$, where Π is a set of sentences stating that every infimum is a minimum, into a counterexample in \mathbf{G}_V to $\Gamma \models A$.

Lemma 5.6. *Let x, \bar{y} be the free variables in A .*

$$\vdash_{\mathbf{H}_0} \forall \bar{y} (\neg \forall x A(x, \bar{y}) \rightarrow \exists x \neg A(x, \bar{y}))$$

PROOF. It is easy to see that in all Gödel logics the following weak form of the law of excluded middle is valid: $\neg \neg A(x) \vee \neg A(x)$. By quantification we obtain

$\forall x \neg \neg A(x) \vee \exists x \neg A(x)$ and, by ISO_0 , $\neg \neg \forall x A(x) \vee \exists \neg A(x)$. Using the intuitionistically valid schema $(\neg A \vee B) \rightarrow (A \rightarrow B)$ we can prove $\neg \forall x A(x) \rightarrow \exists x \neg A(x)$. A final quantification of the free variables concludes the proof. \square

Theorem 5.7. *Let V be an uncountable Gödel set where 0 is isolated. Suppose Γ is a set of closed formulas. Then $\Gamma \models_V A$ iff $\Gamma \vdash_{\mathbf{H}_0} A$.*

PROOF. If: Follows from soundness (Theorem 2.20) and the observation that ISO_0 is valid for any V where 0 is isolated.

Only if: We already know from Theorem 3.9 that the entailment relations of V and $V \cup [\inf P, 1]$ coincide, where P is the perfect kernel of V . So we may assume without loss of generality that V already is of this form, i.e., that $w = \inf P$ and $V \cap [w, 1] = [w, 1]$. Let $V' = [0, 1]$. Define

$$\Pi = \{ \forall \bar{y} (\neg \forall x A(x, \bar{y}) \rightarrow \exists x \neg A(x, \bar{y})) : A(x, \bar{y}) \text{ has } x, \bar{y} \text{ free} \}$$

where $A(x, \bar{y})$ ranges over all formulas with free variables x and \bar{y} . We consider the entailment relation in V' . Either $\Pi, \Gamma \models_{V'} A$ or $\Pi, \Gamma \not\models_{V'} A$. In the former case we know from the strong completeness of \mathbf{H} for $\mathbf{G}_{\mathbb{R}}$ that there are finite subsets Π' and Γ' of Π and Γ , respectively, such that $\Pi', \Gamma' \vdash_{\mathbf{H}} A$. Since all the sentences in Π are provable in \mathbf{H}_0 (see Lemma 5.6) we obtain that $\Gamma' \vdash_{\mathbf{H}_0} A$. In the latter case there is an interpretation \mathcal{J}' such that $\mathcal{J}'(\Pi \cup \Gamma) > \mathcal{J}'(A)$.

It is obvious from the structure of the formulas in Π that their truth value will always be either 0 or 1. Combined with the above we know that for all $B \in \Pi$, $\mathcal{J}'(B) = 1$. Next we define a function $h(x)$ which maps values from $\text{Val}(\mathcal{J}', \Gamma \cup \Pi \cup \{A\})$ into V :

$$h(x) = \begin{cases} 0 & x = 0 \\ w + x/(1-w) & x > 0 \end{cases}$$

We see that h satisfies conditions (1) and (2) of Lemma 2.8, but we cannot use Lemma 2.8 directly, as not all existing infima and suprema are necessarily preserved.

Consider as in Lemma 2.8 the interpretation $\mathcal{J}_h(B) = h(\mathcal{J}'(B))$ for atomic subformulas of $\Gamma \cup \Pi \cup \{A\}$. We want to show that the identity $\mathcal{J}_h(B) = h(\mathcal{J}'(B))$ extends to all subformulas of $\Gamma \cup \Pi \cup \{A\}$. For propositional connectives and the existentially quantified formulas this is obvious. The important case is $\forall x A(x)$. First assume that $\mathcal{J}'(\forall x A(x)) > 0$. Then it is obvious that $\mathcal{J}_h(\forall x A(x)) = h(\mathcal{J}'(\forall x A(x)))$. In the case where $\mathcal{J}'(\forall x A(x)) = 0$ we observe that $A(x)$ contains a free variable and therefore $\neg \forall x A(x) \rightarrow \exists x \neg A(x) \in \Pi$, thus $\mathcal{J}'(\neg \forall x A(x) \rightarrow \exists x \neg A(x)) = 1$. This implies that there is a witness u such that $\mathcal{J}'(A(u)) = 0$. Using the induction hypothesis we know that $\mathcal{J}_h(A(u)) = 0$, too. We obtain that $\mathcal{J}_h(\forall x A(x)) = 0$, concluding the proof.

Thus we have shown that \mathfrak{J}_h is a counterexample to $\Gamma \models_V A$. \square

5.3 0 not isolated but not in the perfect kernel

In the preceding sections, we gave axiomatizations for the logics based on those uncountably infinite Gödel sets V where 0 is either isolated or in the perfect kernel of V . It remains to determine whether logics based on uncountable Gödel sets where 0 is neither isolated nor in the perfect kernel are axiomatizable. The answer in this case is negative. If 0 is not isolated in V , 0 has a countably infinite neighborhood. Furthermore, any sequence $(a_n)_{n \in \mathbb{N}} \rightarrow 0$ is so that, for sufficiently large n , $V \cap [0, a_n]$ is countable and hence, by (the proof of) Theorem 3.8, contains no densely ordered subset. This fact is the basis for the following non-axiomatizability proof, which is a variation on the proof of Theorem 4.3.

Theorem 5.8. *If V is uncountable, 0 is not isolated in V , but not in the perfect kernel of V , then the set of validities of \mathbf{G}_V is not r.e.*

PROOF. We show that for every sentence A there is a sentence A^h s.t. A^h is valid in \mathbf{G}_V iff A is true in every finite (classical) first-order structure.

The definition of A^h mirrors the definition of A^g in the proof of Theorem 4.3, except that the construction there is carried out infinitely many times for $V \cap [0, a_n]$, where $(a_n)_{n \in \mathbb{N}}$ is a strictly descending sequence, $0 < a_n < 1$ for all n , which converges to 0. Let P be a binary and L be a ternary predicate symbol not occurring in A and let R_1, \dots, R_n be all the predicate symbols in A . We use the abbreviations $x \in_n y \equiv \neg \neg L(x, y, n)$ and $x \prec_n y \equiv (P(y, n) \rightarrow P(x, n)) \rightarrow P(y, n)$. As before, for a fixed n , provided $\mathfrak{J}(\exists x P(x, n)) < 1$, $\mathfrak{J}(x \prec_n y) = 1$ iff $\mathfrak{J}(P(x, n)) < \mathfrak{J}(P(y, n))$, and $\mathfrak{J}(x \in_n y)$ is always either 0 or 1. We also need a unary predicate symbol $Q(n)$ to give us the descending sequence $(a_n)_{n \in \mathbb{N}}$: Note that $\mathfrak{J}(\neg \forall n Q(n)) = 1$ iff $\inf\{\mathfrak{J}(Q(d)) : d \in U_{\mathfrak{J}}\} = 0$ and $\mathfrak{J}(\forall n \neg \neg Q(n)) = 1$ iff $0 \notin \{\mathfrak{J}(Q(d)) : d \in U_{\mathfrak{J}}\}$.

Let $A^h \equiv$

$$\left\{ \begin{array}{l} S \wedge \forall n ((Q(n) \rightarrow Q(s(n))) \rightarrow Q(n)) \wedge \\ \neg \forall n Q(n) \wedge \forall n \neg \neg Q(n) \wedge \\ \forall n \forall x ((Q(n) \rightarrow P(x, n)) \rightarrow Q(n)) \wedge \\ \forall n \exists x \exists y (x \in_n 0 \wedge y \in_n 0 \wedge x \prec_n y) \wedge \\ \forall n \forall i [\forall x, y \forall j \forall k \exists z E \vee \forall x \neg (x \in_n s(i))] \end{array} \right\} \rightarrow (A' \vee \exists n \exists u P(u, n) \vee \exists n Q(n))$$

where S is the conjunction of the standard axioms for 0, successor and \leq , with

double negations in front of atomic formulas,

$$E \equiv (j \leq i \wedge x \in_n j \wedge k \leq i \wedge y \in_n k \wedge x \prec_n y) \rightarrow \\ \rightarrow (z \in_n s(i) \wedge x \prec_n z \wedge z \prec_n y)$$

and A' is A where every atomic formula is replaced by its double negation, and all quantifiers are relativized to the predicate $R(n) \equiv \forall i \exists x (x \in_n i)$.

The idea here is that an interpretation \mathfrak{I} will define a sequence $(a_n)_{n \in \mathbb{N}} \rightarrow 0$ by $a_n = \mathfrak{I}(Q(n))$ where $a_n > a_{n+1}$, and $0 < a_n < 1$ for all n . Let $L_n^i = \{x : \mathfrak{I}(x \in_n i)\}$ be the i -th n -level. $P(x, n)$ orders the set $\bigcup_i L_n^i = \{x : \mathfrak{I}(\exists i x \in_n i) = 1\}$ in a subordering of $V \cap [0, a_n]$: $x \prec_n y$ iff $\mathfrak{I}(x \prec_n y) = 1$. Again we force that whenever $x, y \in L_n^i$ with $x \prec_n y$, there is a $z \in L_n^{i+1}$ with $x \prec_n z \prec_n y$, or, if no possible such z exists, $L_n^{i+1} = \emptyset$. Let $r(n)$ be the least i so that L_n^i is empty, or ∞ otherwise. If $r(n) = \infty$ then there is a densely ordered subset of $V \cap [0, a_n]$. So if 0 is not in the perfect kernel, for some sufficiently large L , $r(n) < \infty$ for all $n > L$. $\mathfrak{I}(R(n)) = 1$ iff $r(n) = \infty$ hence $\{n : \mathfrak{I}(R(n)) = 1\}$ is finite whenever the interpretations of P , L , and Q are as intended.

Now if A is classically false in some finite structure \mathfrak{I} , we can again choose a \mathbf{G}_V -interpretation \mathfrak{I}^h so that there are as many n with $\mathfrak{I}^h(R(n)) = 1$ as there are elements in the domain of \mathfrak{I} , and the predicates of A behave on $\{n : \mathfrak{I}(R(n)) = 1\}$ just as they do on \mathfrak{I} .

For instance, we can define \mathfrak{I}^h as follows. We may assume that the domain of $\mathfrak{I} = \{0, \dots, m\}$. Let $U^{\mathfrak{I}^h} = \mathbb{N}$, and $\mathfrak{I}^h(B) = \mathfrak{I}(B)$ for B an atomic subformula of A in the language $\mathcal{L}^{\mathfrak{I}}$. Pick a strictly monotone descending sequence $(a_n)_{n \in \mathbb{N}}$ in V with $\lim a_n = 0$ so that $a_0, \dots, a_{m+1} \in V^\infty$, $a_0 < 1$, $a_{m+1} = \inf V^\infty$, and let $\mathfrak{I}^h(Q(n)) = a_n$. This guarantees that $\mathfrak{I}^h(\forall n ((Q(n) \rightarrow Q(s(n))) \rightarrow Q(n))) = 1$ (because $a_n > a_{n+1}$), $\mathfrak{I}^h(\neg \forall n Q(n)) = 1$ (because $\inf a_n = 0$), $\mathfrak{I}^h(\forall n \neg \neg Q(n)) = 1$ (because $a_n > 0$), and $\mathfrak{I}^h(\exists n Q(n)) < 1$ (because $a_0 < 1$). Then $V \cap [0, a_n]$ is uncountable if $n \leq m$, and countable if $n > m$. For $n \leq m$, let $D_n \subseteq V \cap [0, a_n]$ be countable and densely ordered, and let $j_n : \mathbb{N} \rightarrow D_n$ be bijective.

For $n > m$, let $j_n(0) = a_{n+1}$, and $j_n(i) = 0$ for $i > 0$. Define $\mathfrak{I}^h(P(i, n)) = j_n(i)$. Then, since $j_n(i) < a_n$ for all i , $\mathfrak{I}^h(\forall n \forall x ((Q(n) \rightarrow P(x, n)) \rightarrow Q(n))) = 1$, and, since $j_n(i) < a_0 < 1$, $\mathfrak{I}^h(\exists n \exists u P(u, n)) < 1$. Finally, let $\mathfrak{I}^h(L(x, y, n)) = 1$ for all $x, y \in \mathbb{N}$ if $n \leq m$ (i.e., $L_n^i = \mathbb{N}$), and if $n > m$ let $\mathfrak{I}^h(L(0, 0, n)) = \mathfrak{I}^h(L(1, 0, n)) = 1$ and $\mathfrak{I}^h(L(x, y, n)) = 0$ if $x > 1$ and $y \in \mathbb{N}$, and if $x \in \mathbb{N}$ and $y > 0$ (i.e., $L_n^0 = \{0, 1\}$, $L_n^i = \emptyset$ for $i > m$). This makes the rest of the antecedent of A^h true and ensures that $\mathfrak{I}^h(R(n)) = \mathfrak{I}^h(\forall i \exists x (x \in_n i)) = 1$ if $n \leq m$ and $= 0$ otherwise. Hence $\mathfrak{I}^h(A') = 0$ and $\mathfrak{I}^h \not\models A^h$.

On the other hand, if $\mathfrak{I} \models A^h$, then the value of the consequent is < 1 . Then as required, for all x, n , $\mathfrak{I}(P(x, n)) < 1$ and $\mathfrak{I}(Q(n)) < 1$. Since the antecedent, as

before, must be $= 1$, this means that $x \prec_n y$ expresses a strict ordering of the elements of L_n^i and $\mathfrak{J}((Q(n) \rightarrow Q(s(n))) \rightarrow Q(n)) = 1$ for all n guarantees that $\mathfrak{J}(Q(s(n))) = a_{n+1} < a_n = \mathfrak{J}(Q(n))$. The other conditions are likewise seen to hold as intended, so that we can extract a finite countermodel for A based on the interpretation of the predicate symbols of A on $\{n : \mathfrak{J}(R(n)) = 1\}$, which must be finite. \square

6 Finite Gödel sets

In this section we show that entailment over finite truth value sets are axiomatized by \mathbf{H}_n .

Theorem 6.1. *Suppose Γ contains only closed formulas. Then $\Gamma \models_{V_n} A$ iff $\Gamma \vdash_{\mathbf{H}_n} A$.*

PROOF. If: By Theorem 2.20, since every instance of $\text{FIN}(n)$ is valid in \mathbf{G}_n .

Only if: Suppose $\Gamma \not\models_{\mathbf{H}_n} A$, and consider the set Π of closed formulas of the form

$$\forall \bar{x}_1 \dots \bar{x}_{n-1} ((A_0(\bar{x}_0) \rightarrow A_1(\bar{x}_1)) \vee \dots \vee (A_{n-1}(\bar{x}_{n-1}) \rightarrow A_n(\bar{x}_n))),$$

where A_0, \dots, A_n ranges over all sequences (with repetitions) of length $n+1$ where each A_i is $P(\bar{x})$ for some predicate symbol P occurring in Γ or A . Each formula in Π follows from an instance of $\text{FIN}(n)$ by generalization. Hence, $\Gamma, \Pi \not\models_{\mathbf{H}} A$. From the (strong) completeness (Theorem 5.4) of \mathbf{H} for $\mathbf{G}_{\mathbb{R}}$ we know there is an interpretation $\mathfrak{J}_{\mathbb{R}}$ (into $[0, 1]$) such that $\mathfrak{J}_{\mathbb{R}}(B) = 1$ for all $B \in \Gamma \cup \Pi$ and $\mathfrak{J}_{\mathbb{R}}(A) < 1$.

For sake of brevity let $\text{Val}^a(\mathfrak{J}_{\mathbb{R}}, \Delta)$ for a set of formulas Δ be the set of all truth values of atomic subformulas of formulas in Δ , i.e., $\text{Val}^a(\mathfrak{J}_{\mathbb{R}}, \Delta) = \{\mathfrak{J}_{\mathbb{R}}(P(\bar{u})) : \bar{u} \text{ constants from } \mathcal{L}^{\mathfrak{J}}\}$. We claim that $\text{Val}^a(\mathfrak{J}_{\mathbb{R}}, \Gamma \cup \{A\})$ contains at most n elements. To see this, assume that it contains more than n elements. Then there exist atomic subformulas (w.r.t. \mathfrak{J}) B_0, \dots, B_n of A or of formulas in Γ such that $\mathfrak{J}_{\mathbb{R}}(B_i) > \mathfrak{J}_{\mathbb{R}}(B_{i+1})$ for $i = 0, \dots, n-1$. Thus, $\mathfrak{J}_{\mathbb{R}}((B_0 \rightarrow B_1) \vee \dots \vee (B_{n-1} \rightarrow B_n)) < 1$. But this formula is an instance of a formula in Π , and so we have a contradiction with $\mathfrak{J}_{\mathbb{R}}(B) = 1$.

Now let $\text{Val}^a(\mathfrak{J}_{\mathbb{R}}, \Gamma \cup \{A\}) = \{0, v_1, \dots, v_k, 1\}$ be sorted in increasing order, and let $h(0) = 0, h(1) = 1$, and $h(v_i) = 1 - 1/(i+1)$. Note that any truth value occurring in $\text{Val}(\mathfrak{J}_{\mathbb{R}}, \Gamma \cup \{A\})$ must be one of the elements of $\text{Val}^a(\mathfrak{J}_{\mathbb{R}}, \Gamma \cup \{A\})$. This is easily seen by induction on the complexity of subformulas of $\Gamma \cup \{A\}$ w.r.t. $\mathfrak{J}_{\mathbb{R}}$, as the inf and sup of any subset of the finite set $\text{Val}^a(\mathfrak{J}_{\mathbb{R}}, \Gamma \cup \{A\})$ is a member of the finite set. By Lemma 2.8, \mathfrak{J}_h is a V_n -interpretation with $\mathfrak{J}_h(B) = h(\mathfrak{J}_{\mathbb{R}}) = 1$ for all $B \in \Gamma$ and $\mathfrak{J}_h(A) = h(\mathfrak{J}_{\mathbb{R}}) < 1$. \square

6.1 Remarks on the propositional case and an alternative proof

In the finite case propositional Gödel logics exhibit some interesting properties which we want to mention here.

Proposition 6.2. *If a propositional formula contains less than $n - 2$ variables, then it is valid in propositional \mathbf{G}_n if and only if it is valid in propositional \mathbf{G} (i.e., in LC).*

PROOF. Due to the projective nature of the truth functions, every interpretation of a formula will only need at most n different intermediate values (the number of variables plus 0 and 1 for \perp and \top). \square

Definition 6.3. An n -reduct of a propositional formula A with more than n propositional variables is a formula obtained from A by identifying propositional variables so that the resulting formula contains exactly n variables. If the formula contains less than n propositional variables, the only n -reduct of A is A itself.

It is easy to verify the following propositions:

Proposition 6.4. *If A (propositional) is valid in \mathbf{G}_n , then all n -reducts of A are valid in \mathbf{G}_n .*

PROOF. Every interpretation of an n -reduct of A can be extended to an interpretation of A . \square

Proposition 6.5. *If all n -reducts of A are provable in \mathbf{H} , then A itself is provable in \mathbf{H}_n .*

PROOF. Assume that the n -reducts of A are $\{A_1, \dots, A_N\}$ and assume that for $1 \leq i \leq N$, $\vdash_{\mathbf{H}} A_i$. We assume for simplicity that A contains $n + 1$ variables; let q be one of them and p_1, \dots, p_n the remaining variables (in this case $N = n$). (The general case is proven in the same way.) From $\vdash_{\mathbf{H}} A_i$ we get $\vdash_{\mathbf{H}} (q \leftrightarrow p_i) \rightarrow A$, and from this $(q \leftrightarrow p_i) \vdash_{\mathbf{H}} A$ for every $1 \leq i \leq N$. Thus also

$$\bigvee_{i=1}^N (q \leftrightarrow p_i) \vdash_{\mathbf{H}} A.$$

Together with $\vdash_{\mathbf{H}_n} \bigvee_{i=1}^N (q \leftrightarrow p_i)$ and modus ponens we obtain $\vdash_{\mathbf{H}_n} A$. \square

The notion of n -reducts allows us to show the completeness propositional \mathbf{H}_n with respect to finite-valued propositional Gödel logics.

Theorem 6.6. *A propositional formula is valid in propositional \mathbf{G}_n iff it is provable in propositional \mathbf{H}_n .*

PROOF. The right-to-left direction is done by induction on proofs. If A is valid in \mathbf{G}_n , then all n -reducts of A are valid in \mathbf{G}_n , too (Proposition 6.4). Thus, all the n -reducts are valid in $\mathbf{G}_{\mathbb{R}}$ (Proposition 6.2), and due to the completeness of \mathbf{H} with respect to $\mathbf{G}_{\mathbb{R}}$, all n -reducts are provable in (propositional) \mathbf{H} . Using Proposition 6.5 we obtain that A is provable in \mathbf{H}_n . \square

Together with the Herbrand Theorem for Gödel logics (Theorem 7.8) this allows us to give an alternative proof of Theorem 6.1, although only for the case of weak completeness.

Alternative proof for the weak variant of Theorem 6.1 Observe that in the finite case all quantifier shift rules are valid. Thus, we can transform A into a prenex formula A^* . Later on we will show that for prenex formulas in finite valued logics the Herbrand Theorem holds, i.e., there is a propositional formula B such that A^* is valid iff B is a tautology (Theorem 7.8). Thus A is valid iff the propositional formula B is a tautology. Using the completeness theorem for the propositional case just proven we see that B is provable in propositional \mathbf{H}_n , and thus A is provable in first order \mathbf{H}_n .

Note that this proof only yields weak completeness, i.e., a single formula is valid iff it is provable. \square

7 Fragments

7.1 Prenex fragments

One interesting restriction of the axiomatizability problem is the question whether the prenex fragment of \mathbf{G}_V , i.e., the set of prenex formulas valid in \mathbf{G}_V , is axiomatizable. This is non-trivial, since in general in Gödel logics arbitrary formulas are not equivalent to prenex formulas. Thus, so far the proofs of non-axiomatizability of the logics treated in Sections 4 and 5.3 do not establish the non-axiomatizability of their prenex fragments, nor do they exclude the possibility that the corresponding prenex fragments are r.e. We investigate this question in this section, and show that the prenex fragments of all finite and uncountable Gödel logics are r.e., and that the prenex fragments of all countably infinite Gödel logics are not r.e. The

axiomatizability result is obtained from a version of Herbrand's Theorem for finite and uncountably-valued Gödel logics, which is of independent interest. The non-axiomatizability of countably infinite Gödel logics is obtained as a corollary of Theorem 4.3.

Let V be a Gödel set which is either finite or uncountable. Let \mathbf{G}_V be a Gödel logic with such a truth value set. We show how to effectively associate with each prenex formula A a quantifier-free formula A^* which is a tautology if and only if A is valid in \mathbf{G}_V . The axiomatizability of the prenex fragment of \mathbf{G}_V then follows from the axiomatizability of **LC** (in the infinite-valued case) and propositional \mathbf{G}_n (in the finite-valued case).

Definition 7.1 (Herbrand form). Given a prenex formula $A \equiv Q_1x_1 \dots Q_nx_n B(\bar{x})$ (B quantifier free), the *Herbrand form* A^H of A is $\exists x_{i_1} \dots \exists x_{i_m} B(t_1, \dots, t_n)$, where $\{x_{i_j} : 1 \leq j \leq m\}$ is the set of existentially quantified variables in A , and t_i is x_{i_j} if $i = i_j$, or is $f_i(x_{i_1}, \dots, x_{i_k})$ if x_i is universally quantified and $k = \max\{j : i_j < i\}$. We will write $B(t_1, \dots, t_n)$ as $B^F(x_{i_1}, \dots, x_{i_m})$ if we want to emphasize the free variables.

Lemma 7.2. *If A is prenex and $\mathbf{G}_V \models A$, then $\mathbf{G}_V \models A^H$.*

PROOF. Follows from the usual laws of quantification, which are valid in all Gödel logics. \square

Our next main result will be Herbrand's theorem for \mathbf{G}_V for V uncountable or finite. The *Herbrand universe* $\text{HU}(B^F)$ of B^F is the set of all variable-free terms which can be constructed from the set of function symbols occurring in B^F . To prevent $\text{HU}(B^F)$ from being finite or empty we add a constant and a function symbol of positive arity if no such symbols appear in B^F . The *Herbrand base* $\text{HB}(B^F)$ is the set of atoms constructed from the predicate symbols in B^F and the terms of the Herbrand universe. In the next theorem we will consider the Herbrand universe of a formula $\exists \bar{x} B^F(\bar{x})$. We fix a non-repetitive enumeration C_1, C_2, \dots of $\text{HB}(B^F)$, and let $X_\ell = \{\perp, C_1, \dots, C_\ell, \top\}$ (we may take \top to be a formula which is always = 1). $B^F(\bar{t})$ is an ℓ -instance of $B^F(\bar{x})$ if the atomic subformulas of $B^F(\bar{t})$ are in X_ℓ .

Definition 7.3. An ℓ -constraint is a non-strict linear ordering \preceq of X_ℓ s.t. \perp is minimal and \top is maximal. An interpretation \mathfrak{J} fulfills the constraint \preceq provided for all $C, C' \in X_\ell$, $C \preceq C'$ iff $\mathfrak{J}(C) \leq \mathfrak{J}(C')$. We say that the constraint \preceq' on $X_{\ell+1}$ extends \preceq if for all $C, C' \in X_\ell$, $C \preceq C'$ iff $C \preceq' C'$.

Lemma 2.11 showed that if $h: V \rightarrow W$ is a \mathbf{G} -embedding and \mathfrak{J} is a V -interpretation, then $h(\mathfrak{J}(A)) = \mathfrak{J}_h(A)$ for any formula A . If no quantifiers are involved in A , this also holds without the requirement of continuity. For the following proof we need a similar notion. Let V be a Gödel set, X a set of atomic formulas, and suppose there is an order-preserving, strictly monotone $h: \{\mathfrak{J}(C) : C \in X\} \rightarrow V$ which is

so that $h(1) = 1$ and $h(0) = 0$. Call any such h a *truth value injection on X* . Now suppose B is a quantifier-free formula, and X its set of atomic subformulas. Two interpretations $\mathfrak{I}, \mathfrak{J}$ are *compatible on X* if $\mathfrak{I}(C) \leq \mathfrak{I}(C')$ iff $\mathfrak{J}(C) \leq \mathfrak{J}(C')$ for all $C \in X$.

Proposition 7.4. *Let B^F be a quantifier free formula, and X its set of atomic subformulas together with \top, \perp . If $\mathfrak{I}, \mathfrak{J}$ are compatible on X , then there is a truth value injection h on X with $h(\mathfrak{I}(C^F)) = \mathfrak{J}(C^F)$.*

PROOF. Let $h(\mathfrak{I}(C)) = \mathfrak{J}(C)$ for $B \in X$. Since $\mathfrak{I}, \mathfrak{J}$ are compatible on X , $\mathfrak{I}(C) \leq \mathfrak{I}(C')$ iff $\mathfrak{J}(C) \leq \mathfrak{J}(C')$, and hence $\mathfrak{I}(C) \leq \mathfrak{I}(C')$ iff $h(\mathfrak{I}(C)) \leq h(\mathfrak{I}(C'))$ and h is strictly monotonic. The conditions $h(0) = 0$ and $h(1) = 1$ are satisfied by definition, since $\top, \perp \in X$. We get $h(\mathfrak{I}(B^F)) = \mathfrak{J}(B^F)$ by induction on the complexity of A . \square

Proposition 7.5. (a) *If \preceq' extends \preceq , then every \mathfrak{I} which fulfills \preceq' also fulfills \preceq .* (b) *If $\mathfrak{I}, \mathfrak{J}$ fulfill the ℓ -constraint \preceq , then there is a truth value injection h on X_ℓ with $h(\mathfrak{I}(B^F(\bar{t}))) = \mathfrak{J}(B^F(\bar{t}))$ for all ℓ -instances $B^F(\bar{t})$ of $B^F(\bar{x})$; in particular, $\mathfrak{I}(B^F(\bar{t})) = 1$ iff $\mathfrak{J}(B^F(\bar{t})) = 1$.*

PROOF. (a) Obvious. (b) Follows from Proposition 7.4 together with the observation that \mathfrak{I} and \mathfrak{J} both fulfill \preceq iff they are compatible on X_ℓ . \square

Lemma 7.6. *Let B^F be a quantifier-free formula, and let V be a finite or uncountably infinite Gödel set. If $\mathbf{G}_V \models \exists \bar{x} B^F(\bar{x})$ then there are tuples $\bar{t}_1, \dots, \bar{t}_n$ of terms in $U(B^F)$, such that $\mathbf{G}_V \models \bigvee_{i=1}^n B^F(\bar{t}_i)$.*

PROOF. Suppose first that V is uncountable. By Theorem 3.8, V contains a dense linear subordering. We construct a “semantic tree” \mathbb{T} ; i.e., a systematic representation of all possible order types of interpretations of the atoms C_i in the Herbrand base. \mathbb{T} is a rooted tree whose nodes appear at levels. Each node at level ℓ is labelled with an ℓ -constraint.

\mathbb{T} is constructed in levels as follows: At level 0, the root of \mathbb{T} is labelled with the constraint $\perp < \top$. Let ν be a node added at level ℓ with label \preceq , and let T_ℓ be the set of terms occurring in X_ℓ . Let $(*)$ be: For every interpretation \mathfrak{I} which fulfills \preceq , there is some ℓ -instance $B^F(\bar{t})$ so that $\mathfrak{I}(B^F(\bar{t})) = 1$. If $(*)$ obtains, ν is a leaf node of \mathbb{T} , and no successor nodes are added at level $\ell + 1$.

Note that by Proposition 7.5(b), any two interpretations which fulfill \preceq make the same ℓ -instances of $B^F(\bar{t})$ true; hence ν is a leaf node if and only if there is an ℓ -instance $A(\bar{t})$ s.t. $\mathfrak{I}(A(\bar{t})) = 1$ for all interpretations \mathfrak{I} that fulfill \preceq .

If (*) does not obtain, for each $(\ell + 1)$ -constraint \preceq' extending \preceq we add a successor node v' labelled with \preceq' to v at level $\ell + 1$.

We now have two cases:

(1) T is finite. Let v_1, \dots, v_m be the leaf nodes of T of levels ℓ_1, \dots, ℓ_m , each labelled with a constraint $\preceq_1, \dots, \preceq_m$. By (*), for each j there is an ℓ_j -instance $B^F(\bar{t}_j)$ with $\mathfrak{J}(B^F(\bar{t})) = 1$ for all \mathfrak{J} which fulfill \preceq_j . It is easy to see that every interpretation fulfills at least one of the \preceq_j . Hence, for all \mathfrak{J} , $\mathfrak{J}(B^F(\bar{t}_1) \vee \dots \vee B^F(\bar{t}_m)) = 1$, and so $\mathbf{G}_V \models \bigvee_{i=1}^m B^F(\bar{t}_i)$.

(2) T is infinite. By König's lemma, T has an infinite branch with nodes v_0, v_1, v_2, \dots where v_ℓ is labelled by \preceq_ℓ and is of level ℓ . Each $\preceq_{\ell+1}$ extends \preceq_ℓ , hence we can form $\preceq = \bigcup_\ell \preceq_\ell$. Let $V' \subseteq V$ be a non-trivial densely ordered subset of V , let $V' \ni c < 1$, and let $V'' = V' \cap [0, c)$. V'' is clearly also densely ordered. Now let V_c be $V'' \cup \{0, 1\}$, and let $h : B(A(x)) \cup \{\perp, \top\} \rightarrow V_c$ be an injection which is so that, for all $A_i, A_j \in B(A(x))$, $h(A_i) \leq h(A_j)$ iff $A_i \preceq A_j$, $h(\perp) = 0$ and $h(\top) = 1$. We define an interpretation \mathfrak{J} by: $f^{\mathfrak{J}}(t_1, \dots, t_n) = f(t_1, \dots, t_n)$ for all n -ary function symbols f and $P^{\mathfrak{J}}(t_1, \dots, t_n) = h(P(t_1, \dots, t_n))$ for all n -ary predicate symbols P (clearly then, $\mathfrak{J}(A_i) = h(A_i)$). By definition, \mathfrak{J} ℓ -fulfills \preceq_ℓ for all ℓ . By (*), $\mathfrak{J}(A(\bar{t})) < 1$ for all ℓ -instances $A(\bar{t})$ of $A(x)$, and by the definition of V_c , $\mathfrak{J}(A(\bar{t})) < c$. Since every $A(\bar{t})$ with $\bar{t} \in U(A(x))$ is an ℓ -instance of $A(x)$ for some ℓ , we have $\mathfrak{J}(\exists x A(\bar{x})) \leq c < 1$. This contradicts the assumption that $\mathbf{G}_V \models \exists \bar{x} A(\bar{x})$.

If V is finite, the proof is similar, except simpler. Suppose $|V| = n$. Call a constraint \preceq n -admissible if there is some V -interpretation \mathfrak{J} which fulfills it. Such \preceq have no more than n equivalence classes under the equivalence relation $C \sim C'$ iff $C \preceq C'$ and $C' \preceq C$. In the construction of the semantic tree above, replace each mention of ℓ -constraints by n -admissible ℓ -constraints. The argument in the case where the resulting tree is finite is the same. If T is infinite, then the resulting order $\preceq = \bigcup_\ell \preceq_\ell$ is n -admissible, since all \preceq_ℓ are. Let $c = \max\{b : b \in V, b < 1\}$ and $V_c = V$. The rest of the argument goes through without change. \square

Lemma 7.7. *Let $\exists \bar{x} B^F(\bar{x})$ be the Herbrand form of the prenex formula $A \equiv \bar{Q}_i B(\bar{y}_i)$, and let $\bar{t}_1, \dots, \bar{t}_m$ be tuples of terms in $\text{HU}(B^F)$. If $\mathbf{G}_V \models \bigvee_{i=1}^m B^F(\bar{t}_i)$, then $\mathbf{G}_V \models A$.*

PROOF. For any Gödel set V , the following rules are valid in \mathbf{G}_V :

- (1) $A \vee B \vdash B \vee A$.
- (2) $(A \vee B) \vee C \vdash A \vee (B \vee C)$.
- (3) $A \vee (B \vee B) \vdash A \vee B$.
- (4) $A(y) \vdash \forall x A(x)$.
- (5) $A(t) \vdash \exists x A(x)$.
- (6) $\forall x (A(x) \vee B^{(x)}) \vdash \forall x A(x) \vee B^{(x)}$.
- (7) $\exists x (A(x) \vee B^{(x)}) \vdash \exists x A(x) \vee B^{(x)}$.

The result now follows from Baaz et al. (2001), Lemma 6, and is also easily verified directly. \square

Theorem 7.8. *Let A be prenex, $\exists \bar{x} B^F(\bar{x})$ its Herbrand form, and let V be a finite or uncountably infinite Gödel set. Then $\mathbf{G}_V \models A$ iff there are tuples $\bar{t}_1, \dots, \bar{t}_m$ of terms in $\text{HU}(B^F)$, such that $\mathbf{G}_V \models \bigvee_{i=1}^m B^F(\bar{t}_i)$.*

PROOF. If: This is Lemma 7.7. Only if: By Lemma 7.2 and Lemma 7.6. \square

Remark 7.9. An alternative proof of Herbrand’s theorem can be obtained using the analytic calculus **HIF** (“Hypersequent calculus for Intuitionistic Fuzzy logic”; see Baaz and Zach 2000; Baaz and Ciabattoni 2002).

Theorem 7.10. *The prenex fragment of a Gödel logic based on a truth value set V which is either finite or uncountable infinite is axiomatizable. An axiomatization is given by the standard axioms and rules for **LC** extended by the rules (4)–(7) of the proof of Lemma 7.7. For the n -valued case add $\text{FIN}(n)$.*

PROOF. Completeness: Let $\bar{Q}\bar{y}_i B(\bar{y})$ be a prenex formula valid in \mathbf{G}_V . By Theorem 7.8, a Herbrand disjunction $\bigvee_{i=1}^n B^F(\bar{t}_i)$ is a tautology in \mathbf{G}_V . Hence, it is provable in **LC** or **LC** + $\text{FIN}(n)$. $\bar{Q}\bar{y} B(\bar{y})$ is provable by Lemma 7.7.

Soundness: The rules in the proof of Lemma 7.7 are valid in \mathbf{G}_V . In particular, note that $\forall x(A(x) \vee B) \rightarrow (\forall x A(x) \vee B)$ with x not free in B is valid in all Gödel logics, and $\exists x(A(x) \vee B) \rightarrow (\exists x A(x) \vee B)$ is already intuitionistically valid. \square

In Theorem 4.3, we showed that for every first-order formula A , there is a formula A^g which is valid in \mathbf{G}_V for V countably infinite iff A is valid in every finite classical interpretation. Since finite validity is not r.e., this showed that the set of validities of \mathbf{G}_V is not r.e. We now strengthen this result to show that the prenex fragment of \mathbf{G}_V (for V countably infinite) is likewise not axiomatizable. This is done by showing that there is a formula A^G which is prenex and which is valid in \mathbf{G}_V iff A^g is. Note that not all quantifier shifting rules are generally valid in Gödel logics, so this is a non-obvious result.

Theorem 7.11. *If V is countably infinite, the prenex fragment of \mathbf{G}_V is not r.e.*

PROOF. By the proof of Theorem 4.3, a formula A is true in all finite models iff $\mathbf{G}_V \models A^g$. A^g is of the form $B \rightarrow (A' \vee \exists u P(u))$, where A' is crisp. We show that A^g is validity-equivalent in \mathbf{G}_V to a prenex formula.

From Lemma 4.2 we see that each crisp formula is equivalent to a prenex formula; let A_0 be a prenex form of A' . Since all quantifier shifts for conjunctions are valid, the antecedent B of A^g is equivalent to a prenex formula $Q_1x_1 \dots Q_nx_nB_0(x_1, \dots, x_n)$. Hence, A^g is equivalent to $\bar{Q}\bar{x}B_0(\bar{x}) \rightarrow (A_0 \vee \exists uP(u))$.

Let Q'_i be \exists if Q_i is \forall , and \forall if Q_i is \exists , let $C \equiv A_0 \vee \exists uP(u)$, and $v = \mathcal{I}(\exists uP(u))$. We show that $\bar{Q}\bar{x}B_0(\bar{x}) \rightarrow C$ is equivalent to $\bar{Q}'\bar{x}(B_0(\bar{x}) \rightarrow C)$ by induction on n . Let $\bar{Q}\bar{x}B_0 \equiv Q_1x_1 \dots Q_ix_iB_1(d_1, \dots, d_{i-1}, x_i)$. Since quantifier shifts for \exists in the antecedent of a conditional are valid, we only have to consider the case $Q_i = \forall$. Suppose $\mathcal{I}(\forall x_iB_1(\bar{d}, x_i) \rightarrow C) \neq \mathcal{I}(\exists x_i(B_1(\bar{d}, x_i) \rightarrow C))$. This can only happen if $\mathcal{I}(\forall x_iB_1(\bar{d}, x_i)) = \mathcal{I}(C) < 1$ but $\mathcal{I}(B_1(\bar{d}, c)) > \mathcal{I}(C) \geq v$ for all c . However, it is easy to see by inspecting B that $\mathcal{I}(B_1(\bar{d}, c))$ is either $= 1$ or $\leq v$.

Now we show that $\mathcal{I}(B_0(\bar{d}) \rightarrow (A_0 \vee \exists uP(u))) = \mathcal{I}(\exists u(B_0(\bar{d}) \rightarrow (A_0 \vee P(u))))$. If $\mathcal{I}(A_0) = 1$, then both sides equal $= 1$. If $\mathcal{I}(A_0) = 0$, then $\mathcal{I}(A_0 \vee \exists uP(u)) = v$. The only case where the two sides might differ is if $\mathcal{I}(B_0(\bar{d})) = v$ but $\mathcal{I}(A_0 \vee P(c)) = \mathcal{I}(P(c)) < v$ for all c . But inspection of B_0 shows that $\mathcal{I}(B_0(\bar{e})) = 1$ or $= \mathcal{I}(P(e))$ for some $e \in \bar{d}$ (the only subformulas of $B_0(\bar{d})$ which do not appear negated are of the form $e' \prec e$). Hence, if $\mathcal{I}(B_0(\bar{d})) = v$, then for some e , $\mathcal{I}(P(e)) = v$.

Last we consider the quantifiers in $A_0 \equiv \bar{Q}\bar{y}A_1$. Since A_0 is crisp, $\mathcal{I}(B_0(\bar{d}) \rightarrow (A_0 \vee P(c))) = \mathcal{I}(\bar{Q}\bar{y}(B_0(\bar{d}) \rightarrow (A_1 \vee P(c))))$ for all \bar{d}, c . To see this, first note that shifting quantifiers across \vee , and shifting universal quantifiers out of the consequent of a conditional is always possible. Hence it suffices to consider the case of \exists . $\mathcal{I}(\exists yA_2)$ is either $= 0$ or $= 1$. In the former case, both sides equal $\mathcal{I}(B_0(\bar{d}) \rightarrow P(d))$, in the latter, both sides equal 1. \square

In summary, we obtain the following characterization of axiomatizability of prenex fragments of Gödel logics:

Theorem 7.12. *The prenex fragment of \mathbf{G}_V is r.e. if and only if V is finite or uncountable. The prenex fragments of any two \mathbf{G}_V where V is uncountable coincide.*

7.2 \forall -free fragments

In the following we will denote the \forall -free fragment of \mathbf{G}_V with \mathbf{G}_V^\exists . If Γ, A do not contain \forall and $\Gamma \models_V A$, then $(\Gamma, A) \in \mathbf{G}_V^\exists$.

First we show that for all countably infinite Gödel sets V , the set of validities of \mathbf{G}_V^\exists is not r.e. We will do this by showing that finite validity in classical logic reduces to validity of formulas in \mathbf{G}_V^\exists , specifically, that the formulas A^g used in the proof of Theorem 4.3 are validity-equivalent to \forall -free formulas.

Lemma 7.13. *If $A(x)$ and B are \forall -free, then*

$$\models \forall x A(x) \rightarrow B \quad \text{iff} \quad \models \exists x (A(x) \rightarrow B)$$

PROOF. If: This is a valid quantifier shift rule.

Only if: Suppose that there is an interpretation \mathfrak{J} such that $\mathfrak{J} \not\models \exists x (A(x) \rightarrow B)$, i.e., $\mathfrak{J}(\exists x (A(x) \rightarrow B)) < 1$. This implies that

$$\forall u \in U^{\mathfrak{J}}, \quad \mathfrak{J}(A(u)) > \mathfrak{J}(B). \quad (16)$$

Now define $\mathfrak{J}'(Q)$ for atomic subformulas Q of A by

$$\mathfrak{J}'(Q) = \begin{cases} \mathfrak{J}(Q) & \text{if } \mathfrak{J}(Q) \leq \mathfrak{J}(B) \\ 1 & \text{if } \mathfrak{J}(Q) > \mathfrak{J}(B). \end{cases}$$

Then (i) If C is \forall -free and $\mathfrak{J}(C) > \mathfrak{J}(B)$, then $\mathfrak{J}'(C) = 1$, and if $\mathfrak{J}(C) \leq \mathfrak{J}(B)$, then $\mathfrak{J}'(C) = \mathfrak{J}(C)$; and (ii) $\mathfrak{J}'(\forall x A(x)) = 1$

(i) For atomic C this is the definition of \mathfrak{J}' . The cases for \wedge , \vee , and \rightarrow are trivial. Now let $C \equiv \exists x D(x)$. If $\mathfrak{J}(\exists x D(x)) > \mathfrak{J}(B)$, then for some $u \in U^{\mathfrak{J}}$, $\mathfrak{J}(D(u)) > \mathfrak{J}(B)$. By induction hypothesis, $\mathfrak{J}'(D(u)) = 1$ and hence $\mathfrak{J}'(\exists x D(x)) = 1$. Otherwise, $\mathfrak{J}(\exists x D(x)) \leq \mathfrak{J}(B)$, in which case $\mathfrak{J}'(D(u)) = \mathfrak{J}(D(u))$ for all u . (ii) By (16), for all $u \in U^{\mathfrak{J}}$, $\mathfrak{J}(A(u)) > \mathfrak{J}(B)$, hence, by (i), $\mathfrak{J}'(A(u)) = 1$.

By (i) and (ii) we have that $\mathfrak{J}'(\forall x A(x)) = 1$ and $\mathfrak{J}'(B) = \mathfrak{J}(B) < 1$, thus $\mathfrak{J}'(\forall x A(x) \rightarrow B) < 1$, i.e., $\mathfrak{J}' \not\models \forall x A(x) \rightarrow B$. \square

Note that in the preceding Lemma we can replace the prefix of $A(x)$ by a string of universal quantifiers and the same proof will work.

Lemma 7.14. *If V is a countably infinite Gödel set, then the set of validities of \mathbf{G}_V^{\exists} is not r.e.*

PROOF. It is sufficient to show that the formula A^g used in the proof of Theorem 4.3 is validity-equivalent to a \forall -free formula.

If we only consider the quantifier structure of A^g and apply valid quantifier shifting rules, including the shifting rule for crisp formulas given in Lemma 4.2, we can obtain an equivalent formula which is of the form

$$\forall \bar{x} A(\bar{x}) \rightarrow B$$

where $A(\bar{x})$ and B are \forall -free. By Lemma 7.13, such a formula is validity-equivalent to a \forall -free formula. \square

On the other hand, the \forall -free fragments of all uncountable Gödel sets coincide, and hence are r.e.

Lemma 7.15. *Let $V_1 = [0, 1]$ and V_2 be an uncountable Gödel set. Then $\mathbf{G}_{V_1}^{\exists} = \mathbf{G}_{V_2}^{\exists}$, i.e., $\Gamma \models_{V_1} A$ iff $\Gamma \models_{V_2} A$ provided Γ, A do not contain \forall .*

PROOF. Only if: obvious, since a counter-example in V_2 is also a counter-example in V_1 .

If: Suppose that Γ, A do not contain \forall , and $\Gamma \not\models_{V_1} A$, i.e., there is an \mathfrak{J}_1 such that $\mathfrak{J}(\Gamma) > \mathfrak{J}_1(A)$. Since V_2 is uncountable, its perfect kernel V_2^∞ is non-empty. Let $w = \inf V_2^\infty$ and $V_2' = V_2 \cup [w, 1]$. Let $h: V_1 \rightarrow V_2'$ be $h(v) = w + v/(1-w)$ if $v > 0$ and $= 0$ otherwise. Define \mathfrak{J}_2 for all atomic subformulas B of $\Gamma \cup \{A\}$ by $\mathfrak{J}_2(B) = h(\mathfrak{J}_1(B))$. Then $h' = h \upharpoonright \text{Val}(\mathfrak{J}_1, \Gamma \cup \{A\})$ satisfies the conditions of Lemma 2.8, except the condition requiring existing infs be preserved. However, by Remark 2.9 the definition of \mathfrak{J}_2 still extends to all formulas which do not contain \forall . Since no \forall occur in Γ and A , $\mathfrak{J}_2(\Gamma) > \mathfrak{J}_2(A)$, i.e., $\Gamma \not\models_{V_2'} A$. By Theorem 3.9, $\Gamma \not\models_{V_2} A$. \square

Theorem 7.16. *The \forall -free fragment of \mathbf{G}_V is axiomatizable iff V is finite or uncountable. The \forall -free fragments of any two uncountable Gödel logics coincide.*

PROOF. By Lemma 7.14 and Lemma 7.15. The finite case is obvious as the additional axioms $\text{FIN}(n)$ do not contain universal quantifiers. \square

7.3 \perp -free fragments

In the following we will denote the \perp -free fragment of \mathbf{G}_V with \mathbf{G}_V^\perp . The situation in this case is the same as that in the case of the \forall -free fragments. In fact, the nonaxiomatizability of the \perp -free fragments for countably infinite Gödel sets is a consequence of the nonaxiomatizability of the corresponding \forall -free fragments.

Define A^b as the formula obtained from A by replacing all occurrences of \perp with the new propositional variable b (a 0-place predicate symbol), in particular, any subformula $\neg B$ is replaced by $B \rightarrow b$. Then define A^* as

$$A^* = \left(\bigwedge_{P \in A} \forall \bar{x} (b \rightarrow P(\bar{x})) \right) \rightarrow A^b$$

where $P \in A$ means that P ranges over all predicate symbols occurring in A . We will first prove a lemma relating A^* and A :

Lemma 7.17. *If A does not contain \forall , then $\mathbf{G}_V \models A$ iff $\mathbf{G}_V^\perp \models A^*$.*

PROOF. If: Replace b by \perp in A^* .

Only if: Let $C \equiv \bigwedge_{P \in A} \forall \bar{x}(b \rightarrow P(\bar{x}))$. Now suppose $\mathbf{G}_V^\perp \not\models A^*$. Thus, there is an interpretation \mathfrak{I}_0 such that $\mathfrak{I}_0(A^*) < 1$, i.e., $\mathfrak{I}_0(C) > \mathfrak{I}_0(A^b)$. By Proposition 2.13 we may assume that \mathfrak{I}_0 is countable. Hence, there is a $w \in [0, 1]$, $w \notin \text{Val}(\mathfrak{I}_0, A^*)$, and $\mathfrak{I}_0(A^b) < w < \mathfrak{I}_0(C)$. By Lemma 2.14, there is an interpretation $\mathfrak{I} = \mathfrak{I}_w$ such that $\mathfrak{I}(A^b) < 1$ and $\mathfrak{I}(C) = 1$. Because of the latter, for every atomic subformula B (and hence any subformula of A^b) of A^b , $\mathfrak{I}(B) \geq \mathfrak{I}(b) = v$. Define $\mathfrak{I}'(B)$ for atomic subformulas B of A^b by

$$\mathfrak{I}'(B) = \begin{cases} 0 & \mathfrak{I}(B) = v \\ \mathfrak{I}(B) & \mathfrak{I}(B) > v \end{cases}$$

(and arbitrary for other atomic formulas). It is easily seen by induction that $\mathfrak{I}'(B) = \mathfrak{I}(B)$ if $\mathfrak{I}(B) > v$, and if $\mathfrak{I}(B) = v$, then $\mathfrak{I}'(B) = 0$. In particular, $\mathfrak{I}'(A^b) < 1$. But, of course, $\mathfrak{I}'(b) = \mathfrak{I}'(\perp) = 0$, and hence $\mathfrak{I}'(A^b) = \mathfrak{I}'(A)$. \square

Lemma 7.18. *If V is countably infinite, then the set of validities of \mathbf{G}_V^\perp is not r.e.*

PROOF. By Lemmas 7.14 and 7.17. \square

Lemma 7.19. *Let $V_1 = [0, 1]$ and V_2 be an uncountable Gödel set. Then $\mathbf{G}_{V_1}^\perp = \mathbf{G}_{V_2}^\perp$, i.e., $\Gamma \models_{V_1} A$ iff $\Gamma \models_{V_2} A$, provided Γ, A do not contain \perp .*

PROOF. Exactly analogous to the proof of Lemma 7.15, except taking $h: V_1 \rightarrow V_2'$ to be $h(v) = w + v/(1 - w)$. \square

Theorem 7.20. *The \perp -free fragment of \mathbf{G}_V is axiomatizable iff V is finite or uncountable. The \perp -free fragments of any two uncountable Gödel logics coincide.*

PROOF. By Lemma 7.18 and Lemma 7.19. Again, the \perp -free fragments of finite Gödel logics are axiomatizable since the additional axioms $\text{FIN}(n)$ do not contain \perp . \square

8 Conclusion

In the preceding sections, we have given a complete characterization of the r.e. and non-r.e. first-order Gödel logics. Our main result is that there are two distinct r.e. infinite-valued Gödel logics, viz., $\mathbf{G}_{\mathbb{R}}$ and \mathbf{G}_0 . What we have not done, however, is investigate how many *non-r.e.* Gödel logics there are. It is known that there are continuum-many different propositional consequence relations and continuum-many different propositional quantified Gödel logics (Baaz and Veith, 2000). In forthcoming work (Beckmann et al., 200?), it is shown that there are only countably many first-order Gödel logics (considered as sets of valid formulas). Although this result goes some way toward clarifying the situation, a criterion of identity of Gödel logics using some topological property of the underlying truth value set is a desideratum. We have only given (Lemma 2.11) a sufficient condition: if there is a continuous bijection between V and V' , then $\mathbf{G}_V = \mathbf{G}_{V'}$. But this condition is not necessary: any pair of non-isomorphic uncountable Gödel sets with 0 contained in the perfect kernel provides a quick counterexample (as any two such sets determine $\mathbf{G}_{\mathbb{R}}$ as their logic). A topological characterization of first-order infinite valued Gödel logics could then be used to obtain a more fine-grained analysis of the complexity of the non-r.e. Gödel logics. As noted already, these also differ in the degree to which they are non r.e. (Hájek, 2005).

Another avenue for future research would be to carry out the characterization offered here for extensions of the language. Candidates for such extensions are the addition of the projection modalities ($\Delta a = 1$ if $a = 0$ and $= 0$ if $a < 1$) (Baaz, 1996), of the globalization operator of Takeuti and Titani (1986), or of the involutive negation ($\sim a = 1 - a$). It is known that $\mathbf{G}_{\mathbb{R}}$ with the addition of these operators is still axiomatizable. The presence of the projection modality, in particular, disturbs many of the nice features we have been able to exploit in this paper, for instance, in the presence of Δ the crucial Lemma 2.14 and Proposition 2.15 no longer hold. Thus, not all of our results go through for the extended language and new methods will have to be developed.

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