Decidability of Quantified Propositional Intuitionistic Logic and S4 on Trees of Height and Arity $\leq \omega$ *

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Abstract. Quantified propositional intuitionistic logic is obtained from propositional intuitionistic logic by adding quantifiers $\forall p$, $\exists p$, where the propositional variables range over upward-closed subsets of the set of worlds in a Kripke structure. If the permitted accessibility relations are arbitrary partial orders, the resulting logic is known to be recursively isomorphic to full second-order logic (Kremer, 1997). It is shown that if the Kripke structures are restricted to trees of at height and width at most ω, the resulting logics are decidable. This provides a partial answer to a question by Kremer. The result also transfers to modal **S4** and some Gödel-Dummett logics with quantifiers over propositions.

Keywords: quantified propositional logic, intuitionistic logic, modal logic

1. Introduction

Quantified propositional intuitionistic logic is obtained from propositional intuitionistic logic by adding quantifiers $\forall p$, $\exists p$ over propositions. In the context of Kripke semantics, a proposition is a subset of the worlds in a model structure which is upward closed, i.e., if $h \in P$, then $h' \in P$ for all $h' \geq h$. For propositional intuitionistic logic **H**, several classes of model structures are known to be complete, in particular the class of all partial orders, as well as the class of trees and some of its subclasses. When quantifiers over propositions are added, these results no longer hold. Kremer (1997) has shown that the quantified propositional intuitionistic logic $\mathbf{H}\pi+$ based on the class of all partial orders is recursively isomorphic to full second-order logic. He raised the question of whether the logic resulting from restriction to trees is axiomatizable. The main part of this note establishes that for an important class of trees the resulting logic is, in fact, decidable.

It should be pointed out right away that the trees we consider here are all subtrees of the complete tree of height and arity ω. That is, trees of uncountable arity, or height more than ω are excluded. This is in accord with Kripke's (1965) intuitive interpretation of his possible world semantics for intuitionistic logic. In this interpretation, Kripke explains, the worlds in a structure correspond to "points in time (or 'evidential situations')" and the accessibility relation $\leq$ holds between worlds h, h' if "as far as we know, at time h, we may later gain enough information to advance to h'." If the language is countable,

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then at each point, there are only countably many sentences about which we could discover new information. So at each point, there are only countably many possibilities for advancing to a new evidentiary situation, i.e., the tree of evidentiary situations should have arity $\leq \omega$. Allowing trees of transfinite height would correspond, in this interpretation, to allowing a transfinite process of gathering of evidence. A “jump” to a new evidentiary situation only after an infinite amount of time and investigation seems counter to the spirit of Kripke’s interpretation; hence, trees should be of height at most ω.

The rest of this note is organized as follows: Section 2 introduces the logics considered, and contains several observations regarding the relationship between the classes of formulas valid on various classes of trees. Section 3 presents the decidability result for quantified propositional intuitionistic logic. Section 4 outlines how the results transfer to a proof of decidability of modal **S4** with propositional quantification on similar types of Kripke structures. (Propositionally quantified **S4** on general partial orders is also known to be not axiomatizable.) Intermediate logics based on linear orders (i.e., 1-ary trees), which correspond to Gödel-Dummett logics, are also considered. A concluding section discusses limitations and possible extensions of the method.

2. Quantified propositional intuitionistic logics

DEFINITION 1. An *model structure* $\langle g, K, \leq \rangle$ is given by a set of worlds K, an initial world $g \in K$, and a partial order $\leq$ on K, for which g is a least element. Given a structure, an *(intuitionistic) proposition* is a subset $P \subseteq K$ so that when $h \in P$ and $h' \geq h$, then also $h' \in P$. A *valuation* ϕ is a function mapping the propositional variables to propositions of M. A *model* $M = \langle g, K, \leq, \phi \rangle$ is a structure together with a valuation. If P is a proposition in the model M, then $M[P/p]$ is the model which is just like M except that it assigns the proposition P to p.

DEFINITION 2. If $M = \langle g, K, \leq, \phi \rangle$ is a model, $h \in K$, and A is a formula, we define what it means for *A to be true at h*, denoted $M, h \models A$, by induction on formulas as follows:

1. $M, h \models p$ if $h \in \phi(p)$; $M, h \not\models \bot$.
2. $M, h \models B \wedge C$ if $M, h \models B$ and $M, h \models C$.
3. $M, h \models B \vee C$ if $M, h \models B$ or $M, h \models C$.
4. $M, h \models B \rightarrow C$ if, for all $h' \geq h$, either $M, h' \not\models B$ or $M, h' \models C$.
5. $M, h \models \forall p\, B$, if, for all propositions P, $M[P/p], h \models B$.

6. $M, h \models \exists p B$ if there is a proposition P so that $M[P/p], h \models B$.

The constant $\bot$ is always assigned the empty proposition; $\neg A$ abbreviates $A \to \bot$, hence, $M, h \models \neg B$ iff for all $h' \geq h$, $M, h' \not\models B$.

DEFINITION 3. Given a model M and a formula A, the *proposition defined by A* is the set $M(A) = \{h : M, h \models A\}$.

PROPOSITION 4. *$M(A)$ is a proposition. In fact we have:*

$$\begin{array}{rclrcl} M(p) & = & \phi(p) & M(\bot) & = & \emptyset \\ M(A \wedge B) & = & M(A) \cap M(B) & M(A \vee B) & = & M(A) \cup M(B) \\ M(\forall p A) & = & \bigcap_P M[P/p]A & M(\exists p A) & = & \bigcup_P M[P/p]A \end{array}$$

$$M(A \to B) = \{h : \textit{for all } h' \geq h, \textit{ if } h' \in M(A) \textit{ then } h' \in M(B)\}$$

Proof. By induction on the complexity of formulas. □

DEFINITION 5. A model M *validates* A, $M \models A$, if $M, g \models A$. A model structure S validates A, if every model based on S validates A. A is *valid in a class of model structures* $\mathfrak{C}$, $\mathfrak{C} \models A$, if $M \models A$ for all models M based on structures in $\mathfrak{C}$. A is *valid,* if $M \models A$ for any model M.

DEFINITION 6. A tree T is a subset of ω^*, the set of words over ω, which is closed under initial segments. T is partially ordered by the prefix ordering $\leq$ defined as: $x \leq y$ if $y = xz$ for some z, and totally ordered by the lexicographic order $\preceq$. The empty word Λ is the least element in both orderings. The set $T_\omega = \omega^*$ itself is a tree, the *complete infinitary tree.* The set $T_n = \{i : 0 \leq i < n\}^*$ ($n \leq \omega$) is also a tree (called the *complete n-ary tree*).

DEFINITION 7. We consider the following classes of model structures on trees:

$$\mathfrak{T} = \{\langle \Lambda, T, \leq \rangle : T \text{ is a tree}\} \quad \mathfrak{T}_n = \{T_n\},$$
$$\mathfrak{T}_{\text{fin}} = \{\langle \Lambda, T, \leq \rangle : T \text{ is a finite tree}\}.$$

These models structures give rise to the following quantified propositional logics:

$$\begin{array}{rclrcl} \mathbf{H}\pi+ & = & \{A : \models A\} & \mathbf{Ht}\pi+ & = & \{A : \mathfrak{T} \models A\} \\ \mathbf{Ht}_n\pi+ & = & \{A : \mathfrak{T}_n \models A\} & \mathbf{Ht}^{\text{fin}}\pi+ & = & \{A : \mathfrak{T}_{\text{fin}} \models A\}. \end{array}$$

To each of these quantified propositional logics $\mathbf{L}\pi+$ corresponds a propositional logic $\mathbf{L}$ obtained by restriction to quantifier-free formulas. These all collapse to $\mathbf{H}$, i.e., $\mathbf{H} = \mathbf{Ht} = \mathbf{Ht}_n = \mathbf{Ht}^{\text{fin}}$, for $n \geq 2$ (Gabbay, 1981). The *quantified* propositional logics, however, do not:

PROPOSITION 8. *1.* $\mathbf{H}\pi+ \subsetneq \mathbf{Ht}\pi+ \subsetneq \mathbf{Ht}_n\pi+$ *and* $\mathbf{Ht}\pi+ \subsetneq \mathbf{Ht}^{\text{fin}}\pi+$.
2. $\mathbf{Ht}^{\text{fin}}\pi+ \not\subseteq \mathbf{Ht}_n\pi+$ *and* $\mathbf{Ht}_n\pi+ \not\subseteq \mathbf{Ht}^{\text{fin}}\pi+$.

Proof. The inclusions $\mathbf{H}\pi+ \subseteq \mathbf{Ht}\pi+ \subseteq \mathbf{Ht}_n\pi+$, and $\mathbf{Ht}\pi+ \subseteq \mathbf{Ht}^{\text{fin}}\pi+$ are obvious.

To show that the first inclusion is proper, consider:

$$A \;=\; \forall p(\neg p \vee \neg\neg p) \to \forall p \forall q((p \to q) \vee (q \to p))$$

Then $\mathbf{H}\pi+ \not\models A$: The 4-element diamond is a countermodel. On the other hand, $\mathbf{Ht}\pi+ \models A$, since any h with $h \models \forall p(\neg p \vee \neg\neg p)$ is so that for all $h', h'' \geq h$, either $h' \geq h''$ or $h'' \geq h'$. To see this, suppose $h', h'' \geq h$ but neither $h' \leq h''$ nor $h'' \leq h'$. Consider the proposition $P = \{k : k \geq h'\}$. Then $M[P/p], h' \models p$, and hence $M[P/p], h \not\models \neg p$. On the other hand, $M[P/p], k \not\models p$ for any $k \geq h''$. Hence, $M[P/p], h'' \models \neg p$ and so $M[P/p], h \not\models \neg\neg p$. In other words, the part of the model above h is linearly ordered, and so $h \models \forall p \forall q((p \to q) \vee (q \to p))$.

For the second inclusion, take $B = \forall p(p \vee \neg p)$. Since $\forall p(p \vee \neg p)$ is true at any h which has no successor worlds in a model (a leaf node) and false otherwise, $\neg B$ will be true iff the model has no leaf node. Since complete trees have no leaf nodes, $\mathbf{Ht}_n\pi+ \models \neg B$ but $\mathbf{Ht}\pi+ \not\models \neg B$.[1]

On the other hand, in a finite tree, every branch has a world with no successors. If M is a model based on a finite tree, for every world h there is a world $h' \geq h$ such that $M, h' \models \forall p(p \vee \neg p)$. Hence, for every world h, $M, h \not\models \neg B$ and consequently $M, h \models \neg\neg B$. Thus,

$$\begin{array}{llll} \mathbf{Ht}^{\text{fin}}\pi+ \;\models\; \neg\neg B, \text{ but } & \mathbf{Ht}\pi+, \mathbf{Ht}_n\pi+ & \not\models & \neg\neg B; \\ \mathbf{Ht}^{\text{fin}}\pi+ \;\not\models\; \neg B, \text{ but } & \mathbf{Ht}_n\pi+ & \models & \neg B. \end{array}$$

□

3. Decidability results

THEOREM 9 (Kremer, 1997). $\mathbf{H}\pi+$ *is recursively isomorphic to full second-order logic.*

THEOREM 10. $\mathbf{Ht}\pi+$, $\mathbf{Ht}_n\pi+$, *and* $\mathbf{Ht}^{\text{fin}}\pi+$ *are decidable.*

Proof. We use Rabin's tree theorem (Rabin, 1969). That theorem says that SωS, the monadic second-order theory of T_ω, is decidable. We reduce validity of quantified propositional formulas to truth of formulas of SωS.

The language of SωS contains two relation symbols $\leq$ and $\preceq$, for the prefix ordering and the lexicographical ordering, respectively, and a constant Λ for the empty word. *Finiteness* is definable in SωS: X is finite iff it has a largest element in the lexicographic ordering $\preceq$. Let $x \leq_1 y$ say that y is an immediate successor of x. Then we have:

$$\text{Tree}(T) \;=\; \Lambda \in T \wedge \forall x \in T\, \forall y(y \leq x \to y \in T))$$

[1] This example is due to Tomasz Połacik.

$$\begin{aligned}\mathrm{Prop}(T) &= \forall x \in T\, \forall y (x \leq y \to y \in T)) \\ \mathrm{Arity}_n(T) &= \forall x \in T\, \exists^{=n} y (x \leq_1 y)) \text{ if } n < \omega \\ \mathrm{Fin}(T) &= \exists x \forall y \in T\, y \preceq x)\end{aligned}$$

which say that T is a tree (with root Λ), a proposition, has arity n, and is finite, respectively.

If A is a formula of quantified propositional logic, define A^x by:

$$\begin{aligned}p^x &= x \in X_p & (B \to C)^x &= \forall y \in T (x \leq y \to (B^y \to C^y)) \\ \bot^x &= \bot & (\forall p B)^x &= \forall X_p ((X_p \subseteq T \wedge \mathrm{Prop}(X_p)) \to B^x) \\ (B \wedge C)^x &= B^x \wedge C^x & (\exists p B)^x &= \exists X_p (X_p \subseteq T \wedge \mathrm{Prop}(X_p) \wedge B^x), \\ (B \vee C)^x &= B^x \vee C^x\end{aligned}$$

where y is a new variable not previously used in the translation. Now let

$$\begin{aligned}\Psi(A, \mathbf{Ht}\pi+) &= \forall T (\mathrm{Tree}(T) \to A^x[\Lambda/x]) \\ \Psi(A, \mathbf{Ht}_n\pi+) &= \forall T ((\mathrm{Tree}(T) \wedge \mathrm{Arity}_n(T)) \to A^x[\Lambda/x] \quad (n < \omega) \\ \Psi(A, \mathbf{Ht}^{\mathrm{fin}}\pi+) &= \forall T ((\mathrm{Tree}(T) \wedge \mathrm{Fin}(T)) \to A^x[\Lambda/x]) \\ \Psi(A, \mathbf{Ht}_\omega\pi+) &= \forall T (\forall z (z \in T) \to A^x[\Lambda/x])\end{aligned}$$

We may assume, without loss of generality, that A is closed (no free propositional variables).

We have to show that $\mathrm{S\omega S} \models \Psi(A, \mathbf{L}\pi+)$ iff $\mathbf{L}\pi+ \models A$. First, let $M = \langle \Lambda, K, \leq, \phi \rangle$ be an $\mathbf{L}\pi+$-model (obviously, we may assume that Λ is the root). If $M, \Lambda \not\models A$, then $M(A) \neq K$. Define a variable assignment s for second-order variables by $s(T) = K$. Then it is easy to see that $M(A) = \{x \in K : \mathrm{S\omega S} \models A^x[s]\}$. Thus, $\Psi(A, \mathbf{L}\pi+)$ is false in SωS.

Conversely, if $\mathrm{S\omega S} \not\models \Psi(A, \mathbf{L}\pi+)$, then there is a counterexample witness X for the initial universal quantifier $\forall T$, which is a tree (in the respective class), $\Lambda \in X$, and $\mathrm{S\omega S} \not\models A^x[\Lambda/x][s]$ for $s(T) = X$. (For the case of $\mathbf{L} = \mathbf{Ht}_\omega\pi+$, $X = T_\omega$.)

We show that for any s with $s(T) = X$, the model $M = \langle \Lambda, X, \leq, \phi \rangle$ with $\phi(p) = s(X_p)$ is such that $M(A) = \{x \in X : \mathrm{S\omega S} \models A^x[s]\}$. This is obvious if $A = p$, $A = B \wedge C$ or $A = B \vee C$. Suppose $A = B \to C$. Then $x \in M(A)$ iff for all $y \in X$ with $x \leq y$, $y \notin M(B)$ or $y \in M(C)$. By induction hypothesis, $y \notin M(B)$ iff $\mathrm{S\omega S} \not\models B^y[s]$; similarly for $y \in M(C)$. So $x \in M(A)$ iff $\mathrm{S\omega S} \models A^x[s]$. If $A = \forall p B$, then $x \in M(A)$ iff for all propositions P in X, $x \in M[P/p](B)$. This is the case, by induction hypothesis, iff for all upward-closed subsets P of X, $\mathrm{S\omega S} \models B^x[s']$ where s' is like s except $s'(X_p) = P$; but this is true just in case $\mathrm{S\omega S} \models \forall X_p((X_p \subseteq T \wedge \mathrm{Prop}(X_p)) \to B^x)$. (Similarly for the case of $A = \exists p B$.) Hence, if A is closed and $\mathrm{S\omega S} \not\models \Psi(A, \mathbf{L}\pi+)$, the structure $M = \langle \Lambda, X, \leq, \phi \rangle$ is a countermodel for A. □

4. S4 and Gödel-Dummett logics

Modal logic **S4** is closely related to intuitionistic logic, and its Kripke semantics is likewise based on partially ordered structures and trees. In the modal context, a proposition is any (not necessarily upward-closed) subset of the set of worlds. Adding quantifiers over propositions to **S4**, we obtain the logic **S4**$\pi+$.

Specifically, the semantics of **S4**$\pi+$ is like that for **H**$\pi+$, except that an **S4**-*proposition* in M is a subset $P \subseteq K$, and valuations ϕ map variables to **S4**-propositions. We have the two modal operators $\Box$ and $\Diamond$. $M,h \models A$ is then defined by

1. $M,h \models p$ if $h \in \phi(p)$; $M,h \not\models \bot$.
2. $M,h \models B \wedge C$ if $M,h \models B$ and $M,h \models C$.
3. $M,h \models B \vee C$ if $M,h \models B$ or $M,h \models C$.
4. $M,h \models B \to C$ if $M,h \not\models B$ or $M,h \models C$.
5. $M,h \models \Box B$ if all $h' \geq h$, $M,h' \models B$.
6. $M,h \models \Diamond B$ if some $h' \geq h$, $M,h' \models B$.
7. $h \models \forall p\, B$, if, for all propositions P, $M[P/p],h \models B$.
8. $h \models \exists p\, B$ if there is a proposition P so that $M[P/p],h \models B$.

Depending on the class of Kripke structures considered, we obtain logics **S4**$\pi+$, **S4t**$\pi+$, $\mathbf{S4t}_n\pi+$, $\mathbf{S4t}^{\mathrm{fin}}\pi+$ (for the class of partial orders, trees, n-ary trees, and finite trees, respectively).

The McKinsey-Tarski T-embedding of **H** into **S4** (McKinsey and Tarski, 1948, Theorem 5.1) can be straightforwardly extended to the propositional quantifiers. For a formula A in the language of **H**$\pi+$, define a formula A^T of **S4**$\pi+$ as follows:

$$\begin{array}{rcl@{\qquad}rcl} p^T & = & \Box p & (B \to C)^T & = & \Box(B^T \to C^T) \\ \bot^T & = & \Box\bot & (\forall p B)^T & = & \forall p B^T \\ (B \wedge C)^T & = & B^T \wedge C^T & (\exists p B)^T & = & \exists p B^T \\ (B \vee C)^T & = & B^T \vee C^T & & & \end{array}$$

PROPOSITION 11. **H**$\pi+ \models A$ *iff* **S4**$\pi+ \models A^T$.

Proof. Let $M = \langle g, K, \leq, \phi \rangle$ be an intuitionistic structure, and suppose $M,h \not\models A$. Consider the **S4**-structure $M' = \langle g, K, \leq, \phi' \rangle$ with $\phi'(p) = \phi(p)$. By induction on the complexity of formulas, $M',h \not\models A^T$.

Conversely, if $M' = \langle g, K, \leq, \phi' \rangle$ is an **S4**-structure and $M',h \not\models A^T$, then $M'',h \not\models A^T$, where $M'' = \langle g, K, \leq, \phi'' \rangle$ with $\phi''(p) = M'(\Box p)$. $\Box$

Note that the order structure of M and M' was not changed, so the result holds also relative to any class of tree structures. We can therefore obtain separation results like those in Proposition 8 for the propositionally quantified variants of **S4** by considering the images under the T-embedding of the formulas A, $\neg B$, and $\neg\neg B$ from the proof of Proposition 8.

Fine (1970) and Kremer (1993) showed that $\mathbf{S4}\pi+$, like $\mathbf{H}\pi+$ is not axiomatizable. By the same method used above, the decidability of propositionally quantified versions of **S4** can be established if one is only interested in trees.

PROPOSITION 12. *$\mathbf{S4t}\pi+$, $\mathbf{S4t}_n\pi+$, and $\mathbf{S4t}^{\mathrm{fin}}\pi+$ are decidable.*

Proof. We change the definition of A^x as follows:

$$\begin{array}{rclrcl} p^x & = & x \in X_p & (\Diamond B)^x & = & \exists y(x \leq y \wedge B^y) \\ \bot^x & = & \bot & (\Box B)^x & = & \forall y(x \leq y \rightarrow B^y) \\ (B \wedge C)^x & = & B^x \wedge C^x & \forall p B^x & = & \forall X_p(X_p \subseteq T \rightarrow B^x) \\ (B \vee C)^x & = & B^x \vee C^x & \exists p B^x & = & \exists X_p(X_p \subseteq T \wedge B^x) \\ (B \rightarrow C)^x & = & B^x \rightarrow C^x & & & \end{array}$$

(where y is new.) The definition of $\Psi(A, \mathbf{L}\pi+)$ and the proof that $\mathrm{S}\omega\mathrm{S} \models \Psi(A, \mathbf{L}\pi+)$ iff $\mathbf{L}\pi+ \models A$ (**L** one of $\mathbf{S4t}\pi+$, $\mathbf{S4t}_n\pi+$, $\mathbf{S4t}^{\mathrm{fin}}\pi+$) is the same as for the intuitionistic case, mutatis mutandis. □

Other logics which can be treated using the method used above are Gödel-Dummett logics. These logics were originally characterized as many-valued logics over subsets of $[0,1]$. Here, a *valuation* is a mapping of propositional variables to truth values. A valuation v is extended to formulas by:

$$\begin{array}{rclrcl} v(\bot) & = & 0 & v(A \vee B) & = & \max(v(A), v(B)) \\ v(A \wedge B) & = & \min(v(A), v(B)) & v(A \rightarrow B) & = & \begin{cases} 1 & \text{if } v(A) \leq v(B) \\ v(B) & \text{otherwise} \end{cases} \end{array}$$

In the quantifier-free case, taking any infinite subset of $[0,1]$ as the set of truth values results in the same set of tautologies, axiomatized by $\mathbf{LC} = \mathbf{H} + (A \rightarrow B) \vee (B \rightarrow A)$. This is no longer the case if we add propositional quantifiers. In the many-valued context, these can be introduced by:

$$\begin{array}{rcl} v(\exists p A) & = & \sup\{v[w/p](A) : w \in V\} \\ v(\forall p A) & = & \inf\{v[w/p](A) : w \in V\}, \end{array}$$

where $v[w/p]$ is the valuation which is like v except that it assigns the value w to p. The resulting class of tautologies depends on the order structure of $V \subseteq [0,1]$. In fact, there are $2^{\aleph_0}$ different propositionally quantified Gödel-Dummett logics.

LC is also characterized as the set of formulas valid on the infinite 1-ary tree $\mathfrak{T}_1$. The Gödel-Dummett logic which corresponds to this characterization is $\mathbf{G}_\downarrow\pi$ based on the truth-value set $V_\downarrow = \{0\} \cup \{1/n : n \geq 1\}$, i.e., $\mathbf{G}_\downarrow\pi = \mathbf{Ht}_1\pi+$ (Baaz and Zach, 1998, Proposition 2.8). The intersection of all finite-valued Gödel-Dummett logics, however, coincides with $\mathbf{G}_\uparrow\pi$ with truth value set $V_\uparrow = \{1\} \cup \{1 - 1/n : n \geq 1\}$. Since $\mathbf{G}_\uparrow\pi \neq \mathbf{G}_\downarrow\pi$, this shows that the formulas valid on the infinite 1-ary tree is not identical to the class of formulas valid on all 1-ary trees of finite height. This latter logic was studied and axiomatized by Baaz et al. (2000).

5. Conclusion

The methods used above also show how one can obtain further decidability results for logics based on classes of trees which are definable in the language of SωS. This includes, e.g., tress of finite arity, trees of finite height, and trees of arity or height $\leq n$ for some n.

As argued in the introduction, the notion of trees we consider is the only one which accords with Kripke's intuitive interpretation of intuitionistic model structures. It is nevertheless interesting to consider more general classes of trees (i.e., partial orders with least element and where $h \not\leq h'$ and $h' \not\leq h$ guarantees that for no g is $h, h' \leq g$), or well-founded trees (every branch is well-ordered).

The problem of the complexity of the resulting quantified propositional logics on such structures is the question Kremer (1997) originally raised, and in its general formulation it remains open. The monadic second-order theory of such partial orders is likely not decidable, even if the second-order variables are only allowed to range over upward-closed sets.[2] If it were, the reduction given here would immediately yield the decidability results for the quantified propositional logics on such structures. Undecidability of the monadic theory of all trees of course does not preclude the possibility that quantified propositional intuitionistic logic on all trees is decidable or at least axiomatizable. Indeed, a positive result of this kind would shed light on the expressive power of propositional quantifiers.

Another question we have not touched upon is that of providing "nice" axiomatizations of the logics considered. For 1-ary trees, a quantifier elimination method yields an axiomatization in an extended language (Baaz et al., 2000).

[2] Saharon Shelah, personal communication.

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