Real Options for Managing Risk: Using Simulation to Characterize Gain in Value

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Abstract

This paper explores real options methodology as a risk management tool. Instead of characterizing the value of a real option to an organization as some potentially unrealizable notional market value, it characterizes the real option as a tool for mitigating downside risk while allowing most of the upside potential of a project to flow through to its owner. We do this by simulating the value generated by a real option strategy and comparing it to the alternative strategies of immediate development (based on the NPV rule) and delay as long as possible (generating a european call option).

We compare the cumulative distributions of simulated value for the three strategies and compare them by means of total dominance, and second degree stochastic dominance. Real options do not totally dominate, nor do they always dominate the other two strategies in the second degree. However, the analyst can examine the graphs of the cumulative distributions to see what sort of risk-averse utility function would be needed to justify a preference of one of the alternatives to a real option strategy.

The simulations are performed both for a risk-neutral distribution and for a risk-averse distribution. The distributions differ from each other by a risk premium in the drift of underlying asset value. Strictly speaking, stochastic dominance analysis should be performed on the true (risk-averse) distribution, rather than the risk-neutral distribution. Thus, dominance analysis performed on the risk-neutral distribution implicitly assumes there is no risk premium.

1 Introduction

There are three common methods of accounting for risk in capital budgeting and valuation of real assets.

The most popular textbook method is to calculate a cost of capital that adjusts for risk. In general, the discount rate is adjusted for a risk premium that depends on a measure of systematic risk (β) and price of risk reduction (λ). These risk premia can be measured with the capital asset pricing model (CAPM), consumption CAPM or arbitrage pricing theory (APT). Alternatively, expected cash flows can be adjusted by a risk premium calculated from these models and the result discounted at the riskless rate of return to get a certainty-equivalent model. These methods consider single point measures of value (or net present value) and typically only characterize risk by the second moment (or co-moment with a systematic risk variable) of the project cash flows. This can be motivated by arbitrage considerations, which show that in general, there is some random variable against which the co-moments sufficiently measure the risk premium in market valuation. Capital budgeting decision-making simply comes down to comparing the value of a project to its cost and proceeding if the net benefit is positive.

Another widely advocated approach is to use Monte Carlo simulation[1] of cash flows and project values to assess the total risk profile of the project. The analyst can see the whole distribution of values, including means, high-order moments, medians, and other quantiles. Capital budgeting decisions are more ad hoc in this situation, since there is little theory to guide the analyst as to an appropriate tradeoff between mean payoff and risk measures such as variance of payoff or probability of loss. The analyst could assign a utility function to the payoffs and compare the expected utility of the project to the utility of the investment, but it is harder to identify a utility function if there is a separation of ownership and control (e.g. multiple owners). However, the theory of stochastic dominance can provide an incomplete ordering of various investment projects and provide some guidance in capital budgeting. This approach works well if the utility function is separable and allows consideration of the projects under consideration in isolation. For example, if the project represents almost all of the risky wealth of its owners, the utility of the risky cash flow stream can be used in capital budgeting decisions. This may fail to work if the project is not separable from other sources of wealth because of hedging effects from a correlation between the project and existing wealth or because the mere presence of risky wealth means that the derived utility for incremental wealth coming from another project does not necessarily satisfy the von Neumann Morgenstern assumptions.[2]

[1] The phrase "Monte Carlo simulation" has narrow meanings within the mathematics and computer science literature. Here we use the broad meaning used in finance, in which input variables are simulated using a pseudo-random number generator and the output is examined.

[2] Dybvig and Ross [1] studied portfolio efficient sets, which are sets of portfolios (or firms or projects) that are not second-degree dominated. That is, an efficient portfolio is one that would be chosen by some risk-averse expected-utility-maximizing individual. A linear combination

The third approach to capital budgeting is based on the dynamic changes in the value of a project. Real option value arises because there may be an increase in project value arising from delay. By delaying the project, information can be acquired and risk resolved. Delay may be optimal if the underlying asset value drifts upward at a rate that exceeds the discount rate. However, if the rate of upward drift (adjusted for a risk premium) is less than the the discount rate, there is convenience value in developing the project early, which must be traded off against the risk-reduction benefits of delay. Real option analysis assesses this optimal delay by using a decision tree (lattice) or some other dynamic programming approach. Real option analysis can also utilize the risk-return models of the first approach by assessing certainty-equivalents of the various policies. The certainty-equivalents are typically calculated by adjusting the true probability distribution for a risk premium to get a "risk-neutral" distribution. Risk-neutral expectations are certainty-equivalents.[3]

It is generally agreed that real option analysis gives the optimal solution to a capital budgeting problem, and calculates the best estimate of asset value when real options are present. However, real option analysis merely gives a point estimate of project value and a description of optimal exercise strategy over the time-state space. It does not provide the analyst with other assessments of risk that they would get from a simulation analysis. Analysts are quite accustomed to using simulation as a tool for sensitivity analysis, in part because they are uncertain about the parameters of the underlying process and want to assess the likelihood of their decision resulting in bad outcomes. The purpose of this paper is to provide a simulation analysis of real (american) option decisions, in comparison to alternative benchmarks such as NPV-based decision rules (immediate development if NPV > 0) or rules to delay to the last possible minute (exercising as in a european option strategy).

2 The Model

In this model we compare outcomes of three decision policies for a basic project adoption problem.

Suppose there is a project that we have the option to develop. We can

of two efficient portfolios need not be efficient itself—it could be dominated by some other portfolio. Thus, if an investor chooses an optimal portfolio on the efficient frontier and then considers whether to add to it a new portfolio or project, making the second decision based on stochastic dominance will not necessarily result in an undominated or efficient overall portfolio. Since the resulting portfolio is not efficient, the project may not be optimal. However, for broad classes of utility functions, such as those in the HARA class, these decisions are separable if the project is independent of existing wealth. Moreover, if there is a separation of ownership and control, the analyst will not generally have access to information about the utility functions of the firm owners or the characteristics of the distributions of their risky wealth, so a truly optimal decision cannot be made. A plausible approximation to the optimal decision is that which assumes separation, so that stochastic dominance can be a useful criterion.

[3]The certainty-equivalent of a random variable to be observed in the future is also the forward price for that random variable. This equivalence of risk-neutral expectations, certainty-equivalents and forward prices is straightforward, but rarely highlighted.

choose to develop the project at any time between the present and N time periods in the future. When we develop the project we are required to pay a fixed *development cost* (or *exercise price*) K. Upon payment of the exercise price, the project starts to pay a *dividend yield* on an underlying asset S at a constant rate δ.

The underlying asset has a random value which, over every time period, will go up by a factor of u with probability π or down by a factor $d = 1/u$ with probability $1 - \pi$.

To express the residual value of the project at some future time N, and assuming the development option has been exercised, we will include the random payout S_N, which is the value of the underlying asset at time N.

The problem is to decide when to exercise the option. One policy is to develop the project immediately — the *immediate development policy*. The net present value (NPV) of this policy is $S_0 - K$ where S_0 is the value of the underlying asset at the beginning.

Another policy is the *european real option policy* which only allows development at time N. Given a realization of the model, the NPV of this policy is $\max(S_N - K, 0)$ discounted back to the present at the riskless rate of return.

The optimal policy will depend on both time and the value of the underlying asset value at that time. We represent all possible states for the underlying asset value as a recombining tree with N levels, which is often called a binomial lattice.[4] We can find the optimal policy by computing the corresponding *decision tree* which tells us whether or not we should exercise the option given that we have reached that node in the tree. The optimal policy at the leaves of the tree (representing the last time period) is known. The optimal decisions are recursively determined at the other nodes by a backward process (known as *folding back*). The policy generated by this procedure will be called the *american real option policy*.

In the last section we will investigate the performance of these three policies by simulating the evolution of the underlying asset value. We will compare the expected NPV of the policies as well as comparing the distributions of the net future values of the policies. We will use the risk neutral probability in the simulation, as well as a 'real' probability π which is adjusted by a factor corresponding to a market cost of risk assumption.

3 Comparing Distributions

Consider two random variables X and Y which represent real-valued terminal money payoffs for a given venture. We wish to distinguish the better of the two

[4] On the first level of a recombining tree there is a single node labeled S_0 which will also be called the root of the tree. It has two edges pointing to the two nodes on the next level of the tree labelled uS_0 and dS_0. The node labeled uS_0 points to two nodes on the next level of the tree labeled u^2S_0 and udS_0. The node labeled dS_0 points to two nodes on the next level labeled d^2S_0 and duS_0. Note that since $udS_0 = duS_0$ there are exactly three nodes on the third level. Continuing we define all the N levels of the graph, were level n has exactly n nodes corresponding to the possible values the underlying asset S can have at time n.

outcomes without making any additional assumptions about (market) tradeoffs between risk and expected payoff.

First we have the notion of *total dominance* (or *set dominance*).[5]

Definition 3.1 *For any random variables X and Y, we say that X* totally dominates Y *if $X(s) \geq Y(s)$ for every s in the state space and $X(s) > Y(s)$ for at least one s. We write $X >_T Y$.*

A weaker condition than total dominance is the *first degree stochastic dominance* of Hanoch and Levy [2] which simply expresses an investor's preference for more wealth to less wealth.

Definition 3.2 *For any random variables X and Y, we say X* stochastically dominates Y in the first degree *if any investor with a non–decreasing utility function prefers X to Y and we write $X >_1 Y$. That is, for every non-decreasing utility function u,*

$$\int F_X(x)\, du(x) > \int F_Y(x)\, du(x)$$

on the relevant domain, where F_X and F_Y are the cumulative distribution functions (CDFs) of X and Y.

Hanoch and Levy [2] give a simple characterization of *first degree stochastic dominance.*

Theorem 3.1 *Given random variables X and Y with CDFs F_X and F_Y, then $X >_1 Y$ if and only if $F_X(x) \leq F_Y(x)$ for all x in the domain and there exists at least one x in the domain such that $F_X(x) < F_Y(x)$.*

A third and even weaker condition is that of *second degree stochastic dominance.*

Definition 3.3 *Given random variables X and Y then we say X* stochastically dominates Y in the second degree *if every investor with a non-decreasing concave utility function prefers X to Y (in same sense as above) and we write $X >_2 Y$.*

Note that second degree stochastic dominance precisely expresses the preferences of an investor who would prefer more wealth to less wealth and less risk to more risk.

We will write $X <> Y$ and say X is *incomparable* to Y if none of the above definitions apply. In this case, the decision of X or Y will depend on the level of risk aversion of each individual investor.

What follows describes practical methods for testing for second degree stochastic dominance. The less technical reader may wish to skip ahead to the next section.

Hanoch and Levy [2] give the following characterization of second degree stochastic dominance in the particular case when the CDFs of the random variables have the single–crossing property.

[5]Ingersoll [3] simply refers to this as dominance.

Theorem 3.2 *Given two random variables X and Y with CDFs F_X and F_Y and means μ_X and μ_Y such that for some x_0, $F_X(x) \leq F_Y(x)$ for all $x < x_0$ (and $F_X(x_1) < F_Y(x_1)$ for some $x_1 < x_0$) and $F_X(x) \geq F_Y(x)$ for all $x \geq x_0$ then $X >_2 Y$ if and only if $\mu_X \geq \mu_Y$.*

We will need a refinement of Theorem 3.2 which applies to CDFs of random variables which have the multiple-crossing property.

Definition 3.4 *Given two continuous CDFs F and G we say that $[a,b] \subset \mathbf{R}$ is an* increasing intersection interval for the (ordered) pair (F,G) *if there exists an $\epsilon > 0$ such that*

$$\begin{array}{lll} F < G & \text{on} & [a-\epsilon, a) \\ F = G & \text{on} & [a,b] \\ F > G & \text{on} & (b, b+\epsilon]. \end{array}$$

This is illustrated in Figure 1. We will say $[a,b]$ is a decreasing intersection interval for (F,G) *if $[a,b]$ is an increasing intersection interval for (G,F). That is, there exists an $\epsilon > 0$ such that*

$$\begin{array}{lll} F > G & \text{on} & [a-\epsilon, a) \\ F = G & \text{on} & [a,b] \\ F < G & \text{on} & (b, b+\epsilon]. \end{array}$$

Note that this definition allows a degenerate interval with $a = b$.

Theorem 3.3 *Given random variables X and Y with CDFs F_X and F_Y such that (F_X, F_Y) admits finitely many intersection intervals:*

$$[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n]$$

with

$$a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n$$

denote by

$$\begin{aligned} \alpha_0 &= \int_{-\infty}^{a_1} |F_Y(x) - F_X(x)|\, dx \\ \alpha_i &= \int_{b_i}^{a_{i+1}} |F_Y(x) - F_X(x)|\, dx \text{ for } 1 \leq i \leq n-1 \\ \alpha_n &= \int_{b_n}^{\infty} |F_Y(x) - F_X(x)|\, dx. \end{aligned}$$

If $[a_1, b_1]$ is an increasing intersection interval (from which it follows that $[a_i, b_i]$ is an increasing intersection interval if i is odd and $[a_i, b_i]$ is a decreasing intersection interval if i is even) then $X >_2 Y$ if and only if the following system of

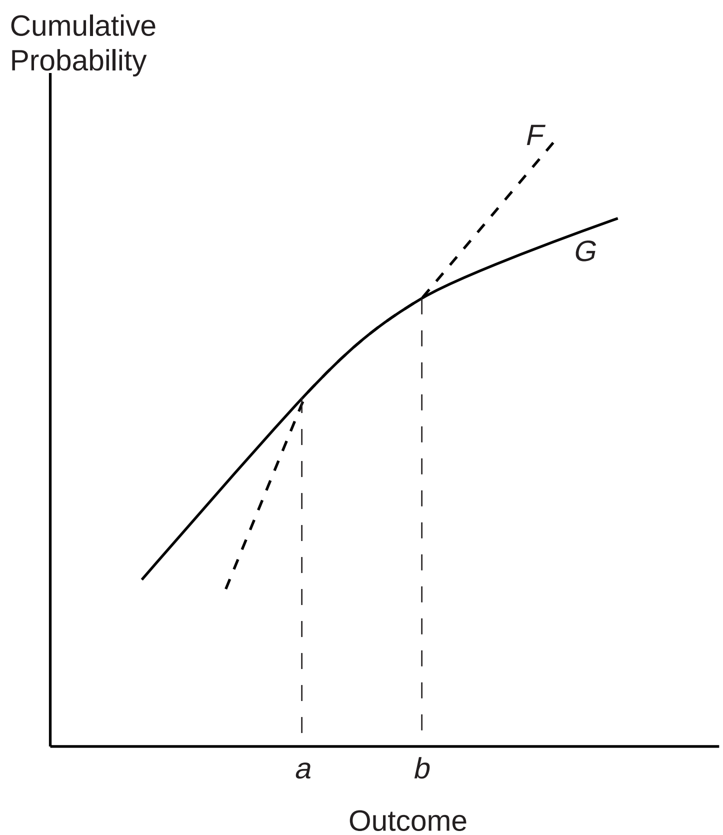

Figure 1: An increasing intersection interval for (F, G).

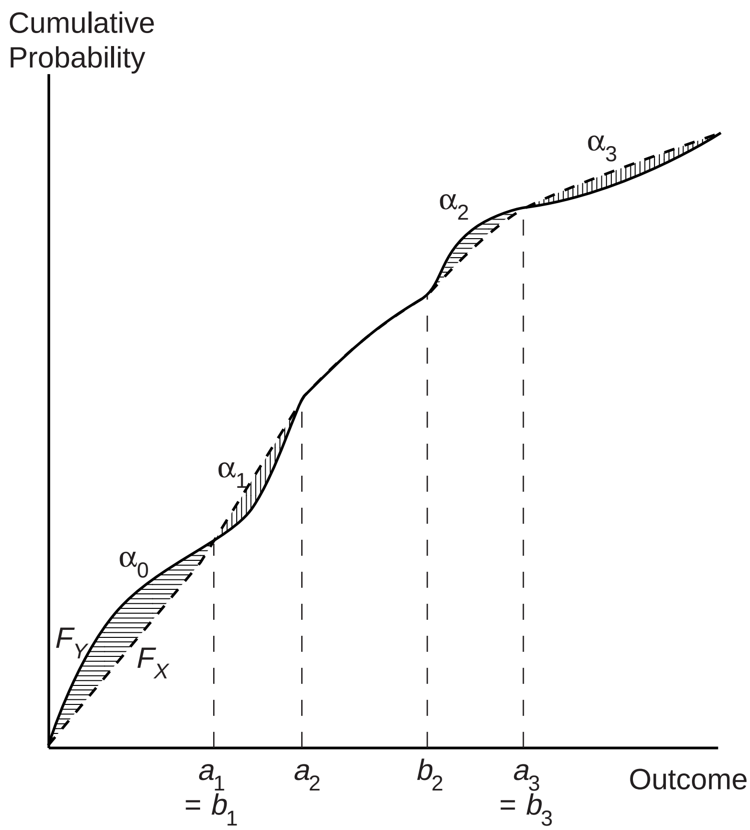

Figure 2: Multiple intersection intervals for the CDFs F_X and F_Y.

inequalities is satisfied (with at least one of the inequalities being strict)

$$\begin{array}{rcl} \alpha_0 & \geq & \alpha_1 \\ \alpha_0+\alpha_2 & \geq & \alpha_1+\alpha_3 \\ \alpha_0+\alpha_2+\alpha_4 & \geq & \alpha_1+\alpha_3+\alpha_5 \\ & \vdots & \\ \alpha_0+\alpha_2+\cdots+\alpha_{2\lfloor\frac{n-1}{2}\rfloor} & \geq & \alpha_1+\alpha_3+\cdots+\alpha_{2\lfloor\frac{n-1}{2}\rfloor+1} \end{array}$$

where for any real number x, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

This is illustrated in Figure 2.

This result will facilitate practical comparisons between distributions whose CDFs cross more than once. The continuity restriction for the CDFs is not essential — it is only meant to ensure that $F_X - F_Y$ changes sign exactly n times. The proof of Theorem 3.3 follows from the next result by Hanoch and Levy.

Theorem 3.4 *Given random variables X and Y with CDFs F_X and F_Y. Then $X >_2 Y$ if and only if*

$$\int_{-\infty}^{t} (F_Y(x) - F_X(x))\, dx \geq 0$$

for every $t \in \mathbf{R}$, with the inequality being strict for at least one value of t.

Note that in the statements that follow, the degree of stochastic dominance reported is the degree of dominance observed by the simulation.

4 Simulation Results

To compare policies we ran simulations of 1040 time steps (representing weekly steps in a 20 year time frame). We fixed the input parameters

$$\begin{array}{rr} S_0 = & 100 \\ K = & 80 \end{array}$$

We set $u = 1/d = 1.036$ which compounds to an annual volatility of 26%. The weekly riskless rate of return is $r = 0.001$ which compounds to a annual rate of 5.33%.

We varied the dividend yield δ. If $\delta = 0$ then it is easy to see that there can be no benefit for early execution and hence in this case the american real option is identical as the european real option. Using the bisection method, we determined that if $\delta > 0.003091$ (17% annually) then the american real option policy advises immediate development and it will be identical to the immediate development policy. Hence, we are interested in behaviour for dividend yields between these two numbers.

For each dividend yield we consider, we ran two simulations. One simulates the dynamics of the underlying asset using the risk-neutral probability $\hat{\pi}$, computed by the formula[6]

$$\hat{\pi} = \frac{\frac{1+r}{1+\delta} - d}{u - d}.$$

The other simulation uses a value π for the true probability which is calculated as $\hat{\pi}$ plus an annual risk premium of 8%. That is

$$\pi = \hat{\pi} + \frac{\sqrt[52]{1.08} - 1}{u - d}.$$

Each simulation consists of 100,000 independent runs. The code was written in C and we used Marsaglia's subtract-with-borrow (pseudo) random number generator (RNG) bit mixed with the Weyl generator[7].

We ran six simulations in total. As we reported earlier, the interesting range of annual yields is between 0 and 17 percent. Experience showed that, for annual yields near the upper end of this interval, the american and the immediate development policy were nearly identical and hence uninteresting to compare. We chose

Low annual dividend yield:	2.03%
Moderate annual dividend yield:	4.10%
High annual dividend yield:	8.36%

For each simulation we represent graphically the cumulative distribution functions (CDF) for the simulated net future payoffs of the three policies we want to compare. To construct the graph, we sorted the present value[8] (discounted at the riskless rate of interest) of the payoff for each strategy by value and then recorded each 100th value, to get 1000 points. We then deleted right tail values the plots lied in intervals $[-\$200, \$1000]$ or $[-\$1000, \$5000]$ as shown. This never resulted in deleting beyond the upper 2% tail. In effect, we are plotting bins of equal probability size, rather than bins of equal payoff width.

[6]This gives a risk-neutral expected capital growth rate of $\frac{1+r}{1+\delta}$ for the underlying asset. With dividends, this gives risk neutral expected rate of return of $1 + r$. See, for example, Sick [6].

[7]We chose this RNG since it has been shown to be at least as good a generator as the standard linear congruential generator but it has a much larger period length, 2^{1407} as opposed to less than 2^{32} (on a 32-bit machine) [4, 5]. The authors feel that the default (or "canned") RNG supplied with many language packages is insufficient given modern day computing power and we suggest that any reader who relies on simulation results should learn about alternate RNGs.

[8]Investor utility and stochastic dominance are based on terminal value. The present value is proportional to terminal value, but also allows a comparison with project NPV.

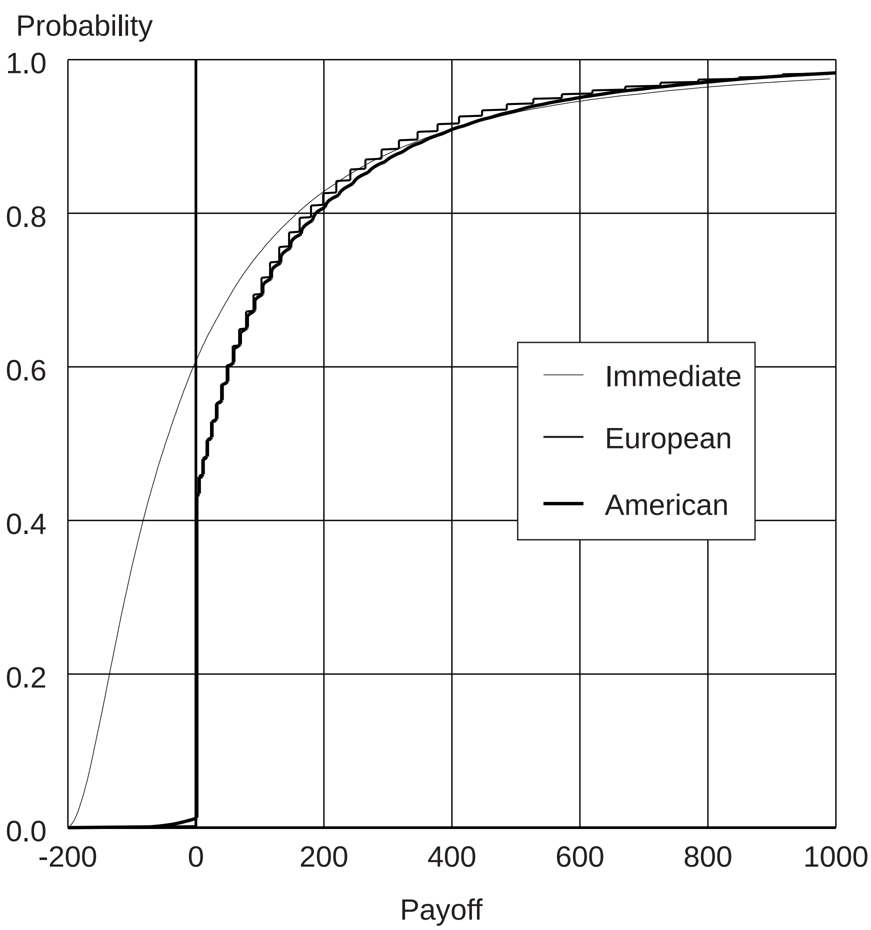

Figure 3: Low dividend, no risk premium.

The european option's CDF will appear jagged since there is a relatively small number of possible outcomes corresponding to the 1040 possible positions at the tips of the tree. Note also that the payoff for the american and immediate development policies can be negative.

4.1 Low Dividend Yield, No Risk Premium

The annual dividend yield is set at 2.03% ($\delta = 0.0003865$). No risk premium is assumed and thus $\hat{\pi} = \pi = 0.4998$.

Policy	**Simulated NPV**	**Theoretical NPV**
American Real Option	\$49.39	\$49.39
Immediate Development	\$20.06	\$20.00
European Real Option	\$44.49	\$44.50

The simulated expected values are close to the theoretical values, suggesting that it provides a good approximation. Comparing the results on each simulation runs across the three policies yields:

X	vs.	Y	X > Y	X = Y	X < Y
American	vs.	Immediate	99.2%	0%	0.8%
American	vs.	European	31.4%	63.5%	16.6%
Immediate	vs.	European	14.0%	0%	86.0%

The american option almost totally dominates the immediate development strategy.

Figure 3 shows the CDFs for the three policies. Note how the american and european options shift the left tail probabilty to the right. This risk-managment outcome allows them to stochastically dominate the immediate development strategy in the second degree:

$$\begin{array}{rcl} \text{American Real Option} & >_2 & \text{Immediate Development} \\ \text{American Real Option} & <> & \text{European Real Option} \\ \text{European Real Option} & >_2 & \text{Immediate Development} \end{array}$$

The presence of stochastic dominance shows that any reasonable investor will choose the american policy over the immediate policy. The graph shows that if the option is "in the money" the american policy outperforms the european policy but there is a 1.3% chance that the american option will have a negative return versus the european policy which has no chance of a negative return. This is the only reason why the american and european policies are incomparable. Only the most risk-averse investor would choose to adopt the european policy in this case.

4.2 Moderate Dividend Yield, No Risk Premium

The annual dividend yield is set at 4.10% ($\delta = 0.0007730$). No risk premium is assumed and thus $\pi = \hat{\pi} = 0.4944$.

Policy	**Simulated NPV**	**Theoretical NPV**
American real option	\$37.24	\$37.09
Immediate Development	\$20.11	\$20.00
European Real Option	\$25.22	\$25.12

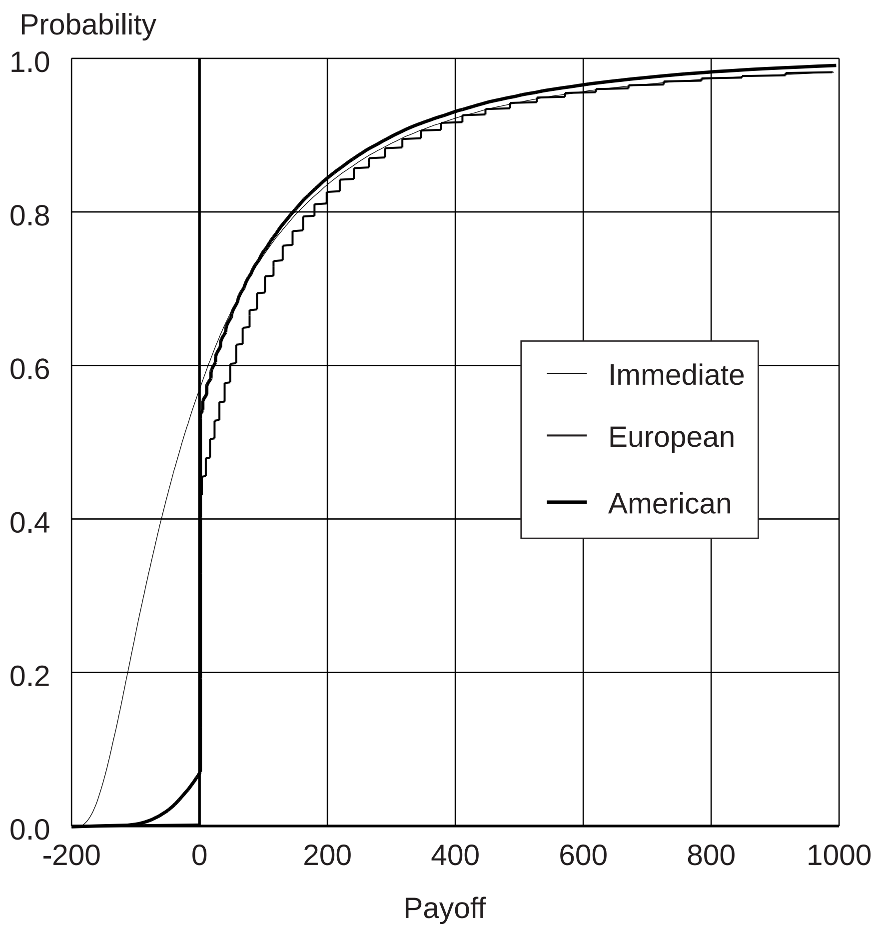

Figure 4: Medium dividend, no risk premium.

X	vs.	Y	X > Y	X = Y	X < Y
American	vs.	Immediate	65.7%	0%	34.3%
American	vs.	European	42.8%	49.5%	7.7%
Immediate	vs.	European	36.4%	0%	63.6%

Figure 4 shows the graphical comparison of the CDFs for the three policies. We have that

American Real Option	$>_2$	Immediate Development
American Real Option	$<>$	European Real Option
European Real Option	$>_2$	Immediate Development

Stochastic dominance again shows that any reasonable investor will choose the american policy over the immediate policy. Once again, the american policy outperforms the european policy but there is a 7% chance that the american real option will have a negative return. Note that the large difference in the simulated NPV suggests that only a risk adverse investor would prefer the european policy over the american.

4.3 High Dividend Yield, No Risk Premium

The annual dividend yield is set at 8.36% ($\delta = 0.0015450$). No risk premium is assumed and thus $\pi = \hat{\pi} = 0.4835$.

Policy	**Simulated NPV**	**Theoretical NPV**
American Real Option	\$25.70	\$25.70
Immediate Development	\$20.04	\$20.00
European Real Option	\$6.70	\$6.75

X	vs.	Y	X > Y	X = Y	X < Y
American	vs.	Immediate	38.4%	0%	61.6%
American	vs.	European	41.1%	42.3%	16.6%
Immediate	vs.	European	49.8%	0%	50.2%

Figure 5 shows the graphical comparison of the CDFs for the three policies. We have that

American Real Option	$>_2$	Immediate Development
American Real Option	$<>$	European Real Option
European Real Option	$<>$	Immediate Development

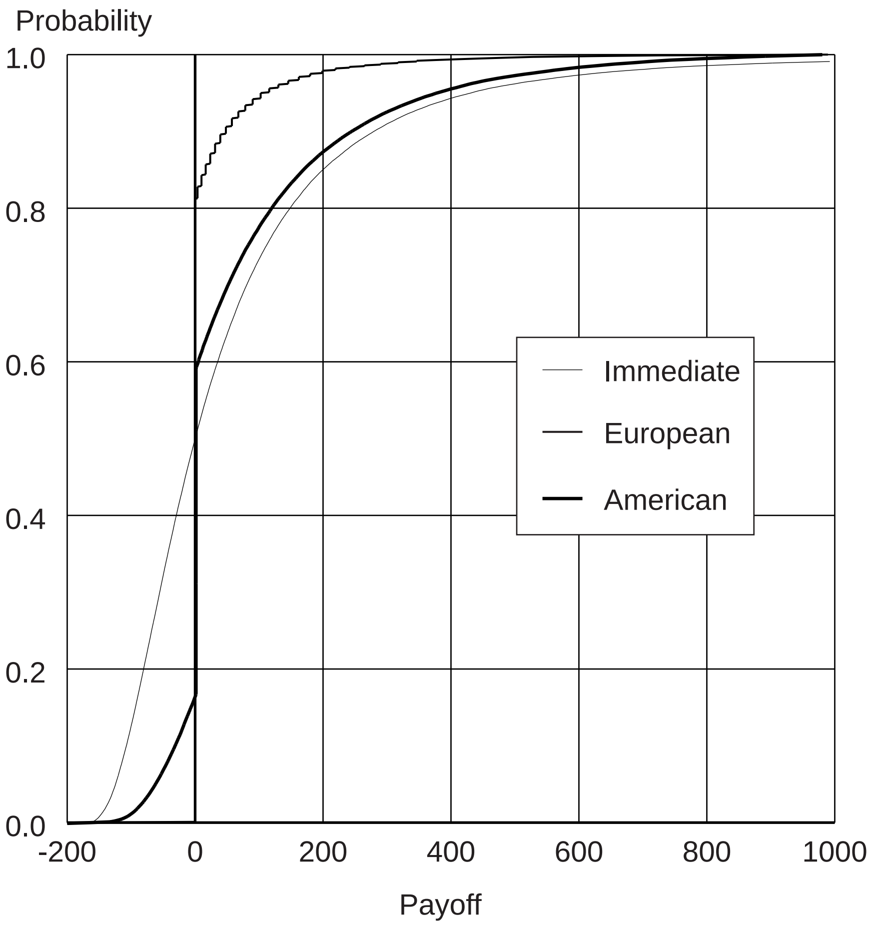

Figure 5: High dividend, no risk premium.

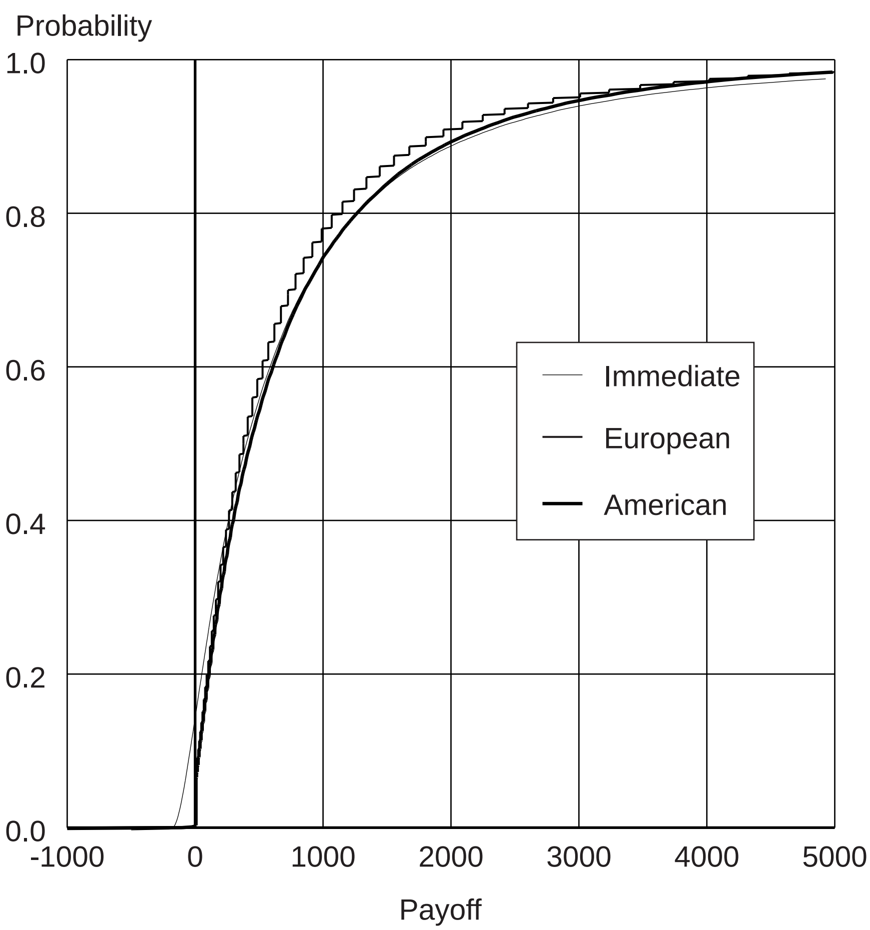

Figure 6: Low dividend, risk premium.

What is interesting here is that, on any given run, the immediate development policy will outperform the american policy 61.6% of the time but the american policy still stochastically dominates the immediate policy. The european policy has a very poor simulated NPV and hence most investors will choose to assume the risk of the american policy.

4.4 Low Dividend Yield, Risk Premium

The annual dividend yield is set at 2.03% ($\delta = 0.0003865$). An 8% annual risk premium is assumed, so that $\hat{\pi} = 0.4998$ is used to find the american policy but the true probability is $\pi = 0.5208$ is used to simulate the underlying asset value.

Policy	**Simulated NPV**	**Theoretical NPV**
American Real Option	\$318.21	\$318.98
Immediate Development	\$305.06	\$306.42
European Real Option	\$282.27	\$283.49

Note that these expcted NPVs are much larger than before because of the upward drift of the risk premia. Thus, for example, the theoretical expected NPV of immediate development is \$306.42, which is larger than the \$20 value of immediate development. This is because we only discounted the future values at the riskless rate of interest. For valuation purposes, we would either have to use a certainty equivalent (as in the risk-neutral expectation given earlier) or use a risk-adjusted discount rate. Since the real option gives real operating leverage that is stochastic, the risk-adjusted discount rate is a random variable and cannot be used in valuation. Note that discounting the intermediate dividends at the riskless rate is equivalent to compounding intermediate dividends forward at the riskless rate to get a future value. An investor receiving these dividends may be able to reinvest them at a higher risk-adjusted expected rate of return, so our approach is a somewhat conservative assessment of projects that are developed early.

X	vs.	Y	$X > Y$	$X = Y$	$X < Y$
American	vs.	Immediate	97.5%	0%	2.5%
American	vs.	European	79.5%	17.5%	3.0%
Immediate	vs.	European	53.7%	0%	46.3%

Figure 6 shows the CDFs for the three policies. We have that

$$\begin{array}{lcl} \text{American Real Option} & >_2 & \text{Immediate Development} \\ \text{American Real Option} & <> & \text{European Real Option} \\ \text{European Real Option} & <> & \text{Immediate Development} \end{array}$$

The presence of stochastic dominance shows that any reasonable investor will choose the american policy over the immediate policy. The graph shows that for middle and high-end returns the american policy outperforms the european policy but there is a 0.2% chance that the real option will have a negative return versus the european policy which has no chance of a negative return. Given the relatively large difference in the simulated NPV for these two policies, an investor would have to be very risk-averse to choose the european policy over the american.

4.5 Moderate Dividend Yield, Risk Premium

The annual dividend yield is set at 4.10% ($\delta = 0.0007730$). An 8% annual risk premium is assumed, so that $\hat{\pi} = 0.4944$ is used to find the american policy but

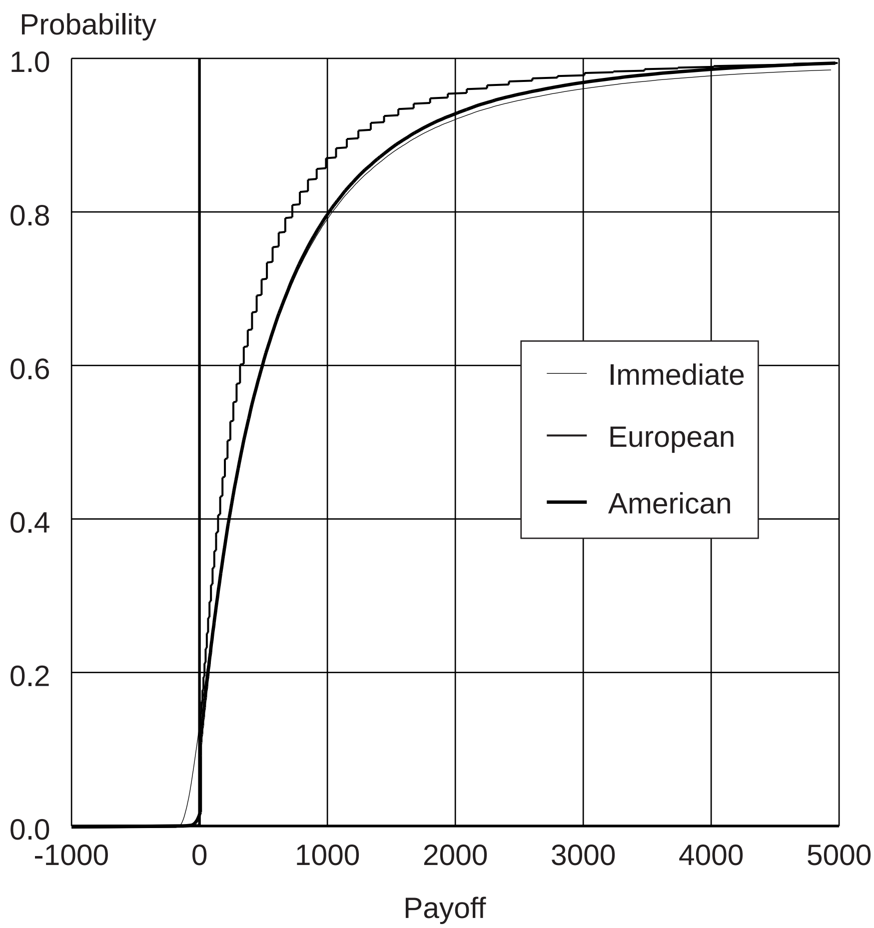

Figure 7: Medium dividend, risk premium.

the true probability $\pi = 0.5153$ is used to simulate the underlying asset value.

Policy	**Simulated NPV**	**Theoretical NPV**
American Real Option	\$249.33	\$248.49
Immediate Development	\$247.54	\$247.39
European Real Option	\$181.89	\$181.38

X	vs.	Y	X > Y	X = Y	X < Y
American	vs.	Immediate	33.5%	0%	66.5%
American	vs.	European	86.9%	10.7%	2.4%
Immediate	vs.	European	79.8%	0%	20.2%

Figure 7 shows the graphical comparison of the CDFs for the three policies. We have that

$$\begin{array}{lcl} \text{American Real Option} & >_2 & \text{Immediate Development} \\ \text{American Real Option} & <> & \text{European Real Option} \\ \text{European Real Option} & <> & \text{Immediate Development} \end{array}$$

Again we have stochastic dominance to insure the preference of the american policy over the immediate policy. Since the american policy has only a 1.8% chance of being negative, most investors would prefer the small burden of risk assumed by choosing the american policy over the european policy in order to gain the large increase in simulated NPV.

4.6 High Dividend Yield, Risk Premium

The annual dividend yield is set at 8.36% ($\delta = 0.0015450$). An 8% annual risk premium is assumed, so that $\hat{\pi} = 0.4944$ is used to find the American policy but the true probability $\pi = 0.5044$ is used to simulate the underlying asset value.

Policy	**Simulated NPV**	**Theoretical NPV**
American Real Option	\$163.75	\$162.14
Immediate Development	\$169.57	\$169.19
European Real Option	\$69.62	\$69.36

X	vs.	Y	X > Y	X = Y	X < Y
American	vs.	Immediate	9.6%	0%	90.4%
American	vs.	European	82.9%	12.5%	4.6%
Immediate	vs.	European	88.0%	0%	12.0%

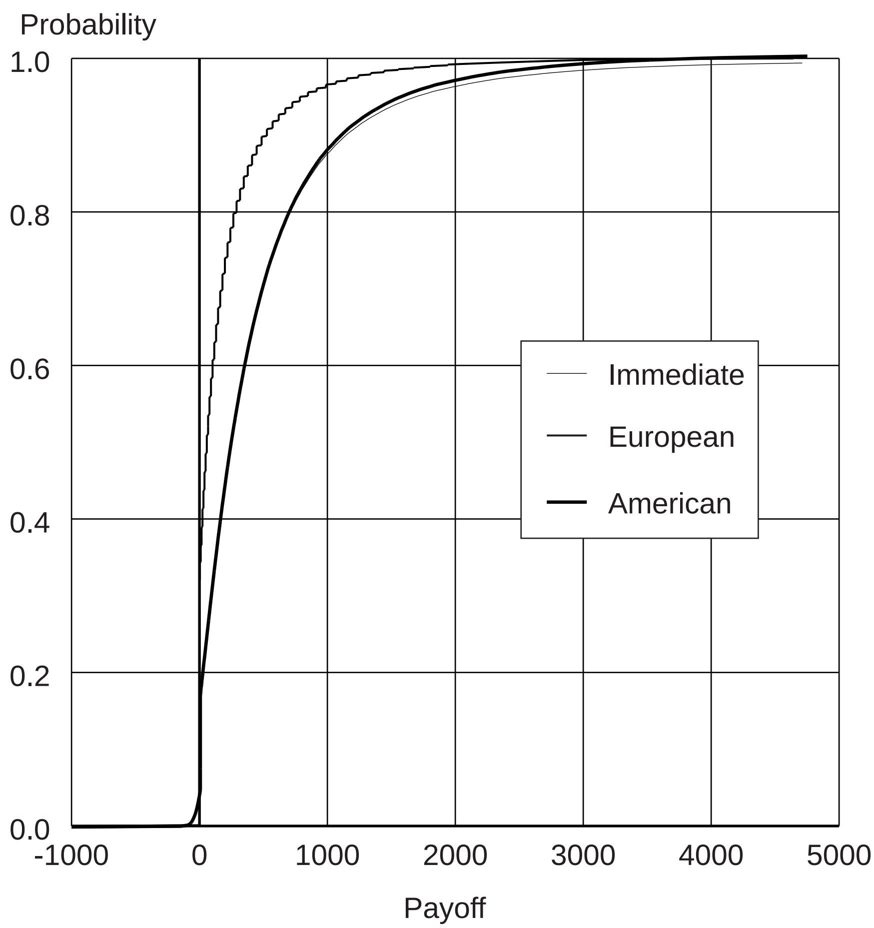

Figure 8: Low dividend, risk premium.

Figure 8 shows the graphical comparison of the CDFs for the three policies. We have that

American Real Option	$<>$	Immediate Development
Aamericanmerican Real Option	$<>$	European Real Option
European Real Option	$<>$	Immediate Development

Analysis in this case is more difficult. Most investors will rule out the european policy due to the low simulated NPV. But notice that in this case the immediate development policy has a slightly higher simulated NPV than the american policy but the american policy does carry less risk. The individual investor will have to decide if the increase in NPV gained by choosing the immediate policy is worth the extra risk assumed.

5 Conclusion

This paper proposes that real options analysts consider using simulation to characterize the risk-management advantages of real options. This allows a manager representing the owner of a real option useful insight into the source of value generated by a real option and some sensitivity analysis for the problem. Many senior managers are not technically oriented, and may be reluctant to accept a single point-estimate of value associated with a real option strategy. This approach may make real options more palatable to these reluctant managers.

By examining the cumulative distribution function of real options strategies and alternative strategies, the decision-maker can determine whether one strategy dominates the other (in the strict sense of second degree stochastic dominance). Alternatively, she may find that the real "almost dominates" another in the sense that it almost has dominance, except for a multiple crossing over a very narrow range of payoffs. This sometimes happened in the case of real options with an american early exercise strategy, in comparison to the immediate exercise exercise strategy.

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