Modelling Electricity Price Risk[1]

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[1]We would like to acknowledge the assistance of Owen Craig at the Power Pool of Alberta for arranging to get us the data and for providing helpful background on the Alberta Pool. The latest version of this paper can be retrieved at http://www.ucalgary.ca/~sick. This is part of our continuing research into electricity, which includes the publication Elliott-Sick-Stein [1]. This paper contains some of the material from that paper for the convenience of the reader and the final draft will be disjoint from it, with a focus on the Expectation Maximization approach to econometrics and the Fourier approach to option valuation. The earlier approach used a two-step econometric approach and a bootstrap simulation approach to the estimation of densities.

Abstract

Modelling Electricity Price Risk

In this paper we develop a general model of spot electricity price that encompasses the stylized features of many of the emerging deregulated electricity pools around the world. We incorporate seasonality on an annual basis and a daily basis around a mean-reverting de-seasonalized intrinsic price.

A unique feature of this paper is the treatment of jumps in the spot price as arising from supply shocks as large generators in the system come off-line and go on-line in a partially predictable manner. We model the number of large generators on line as a discrete Markov process. This feature is motivated by the Alberta electricity pool, which has 14 large base-load generators and very little excess capacity.

We show how to estimate the diffusion process with a Kalman filter technique and the discrete Markov model with maximum likelihood model.

The motivation for pricing calls on this price process is two-fold. First many electricity customers purchase call options to manage their risk. Second, generators are called into the system or turned off according to whether their marginal price is less than or greater than the system marginal price (spot price). The revenue stream to a company that builds a new generator that is not part of base load will be a strip of call options. Thus, this is a real option valuation model.

1 Market Structure

This paper models the Alberta electricity pool as an example of a deregulating supply and demand model. The Alberta market is supplied by 3 large utilities that have some very efficient base load coal generators (using low-sulphur strip mines adjacent to the power plants), plus an array of natural gas plants. There are also some hydro-electric plants and alternative energy plants, but the bulk of the excess demand is supplied by limited-capacity links to adjacent jurisdictions. British Columbia, to the west, has hydro power and Saskatchewan, to the east, has thermal coal power. There is very little peaking capacity in Alberta itself.

For example, the March 11, 2000 Power Pool Report identifies 917 Megawatts of reserve capacity for the Alberta Interconnected Electric System (AIES):

SUMMARY	
INTERCHANGE	-63
AIES NET GEN	5534
AIES TOTAL LOAD	5605
REQUIRED RESERVE	455
GENERATION RESERVE	904
OPERATING RESERVE	917

Of this reserve, most is from hydro (MCR is the Maximum Capacity Rating):

GENERATION GROUP	MCR	NET	RES
COAL	5621	4883	135
GAS	946	285	116
HYDRO	789	50	653
IPP	N/A	316	N/A
TOTAL	7356	5534	904

The interchange is not allowed to count as reserve. In this case it was supplying a small amount of electricity becasue it has a negative quantity:

INTERCHANGE LINE	ACTUAL
BC HYDRO	41
SASKPOWER	-24
MEDICINE HAT	-80
TOTAL	-63

The interchange can provide up to 800 MW of power.

The generation capacity included thermal (primarily coal) generation capacity from three large generation companies (Transalta, Atco and Epcor) plus some indendent power producers (IPP):

TRANSALTA THERMAL UNIT	MCR	NET	RES
KH1	381	370	0
KH2	381	1	0
SD1	279	293	0
SD2	279	275	0
SD3	353	367	0
SD4	353	358	0
SD5	353	349	0
SD6	364	359	0
WB1	64	0	0
WB2	64	52	0
WB3	140	0	0
WB4	279	250	0

ATCO ELEC THERMAL UNIT	MCR	NET	RES
SH1	378	324	56
SH2	378	343	36
BR3	148	135	15
BR4	148	141	13
BR5	368	380	0
HRM	143	125	15
ST1	10	0	0
ST2	8	0	0
RB1	26	0	0
RB2	40	42	0
RB3	21	20	0

EPCOR THERMAL			
UNIT	MCR	NET	RES
GN1	384	376	0
GN2	384	385	0
CB1	158	40	24
CB2	158	60	24
CB3	158	40	24
CB4	158	61	24
RD8	67	0	0
RD9	71	0	0
R10	71	22	20

NEW IPP			
UNIT	MCR	NET	RES
PRM	85	89	0
PH1	47	44	0
RL1	47	0	0
FNG	47	0	0

Peaking power comes from the hydro plants of Transalta plus the British Columbia interconnect, which has a maximum capacity of 800 Megawatts.

TRANSALTA HYDRO			
UNIT	MCR	NET	RES
CAS	34	-2	17
THS	2	0	0
SPR	100	0	100
RUN	47	0	47
INT	5	0	0
POC	14	0	0
BAR	12	0	10
KAN	19	7	0
HSH	15	7	0
GHO	55	1	46
BPW	16	6	0
BOW	319	19	220
BIG	120	37	79
BRA	350	-6	354

The spot price of electricity is set hourly and applies as a pool price throughout the province. There are no differential transmission charges for location. Generators are switched on and off by a System Controller who matches supply and demand by ranking the generators by their bid price into a merit order (supply curve) and setting the spot price or system marginal price (SMP) to the bid price of the last generator needed to meet the demand.[1] Demand is not very price sensitive, but follows seasonal patterns throughout the year, the week, the day and has random weather effects, as well. Supply and demand are summarized on the Power Pool website in real time as shown in Figure 1. Note the inelastic demand curve and the step-function Merit Order for the supply curve. The price scale is logarithmic, so prices can jump sharply, and supply becomes quite inelastic at high quantities. Thus, removing one of the large base-load generators from the bottom of the Merit Order causes the supply curve to shift to the left. If the demand is high, price is determined by the intersection of inelastic supply and demand curves, so a price spike arises.

This suggests that a good model for prices will involve seasonal and random influences on demand coupled with discrete changes in supply. The major supply changes will involve the 14 large baseload generators that can supply over 250 Megawatts of power.

2 Price Process

The spot price $S = \{S_t, t \geq 0\}$ of electricity will be modelled as a one-factor diffusion with jumps in the price described by a finite state Markov chain $\overline{Z} = \left\{\overline{Z}_t, t \geq 0\right\}$. The diffusion is mean reverting with a diffusion level μ_t that depends on $\overline{Z}$. Our processes are defined on $(\Omega, \mathcal{F}, P)$.

The continuous time parameter $t \geq 0$ will represent days. An annual periodic factor in the price will be modelled by

$$f_t := \exp\left[\gamma \int_0^t \sin\left(\frac{2\pi}{365}s + \phi\right) ds\right]. \tag{1}$$

[1] It is possible for the consumers to actually bid to supply electricity. In this case, they offer to shed load when the price rises above a given point. In this case, the last offer (needed to generate electricity to meet demand) that sets the SMP might be from a consumer rather than a producer.

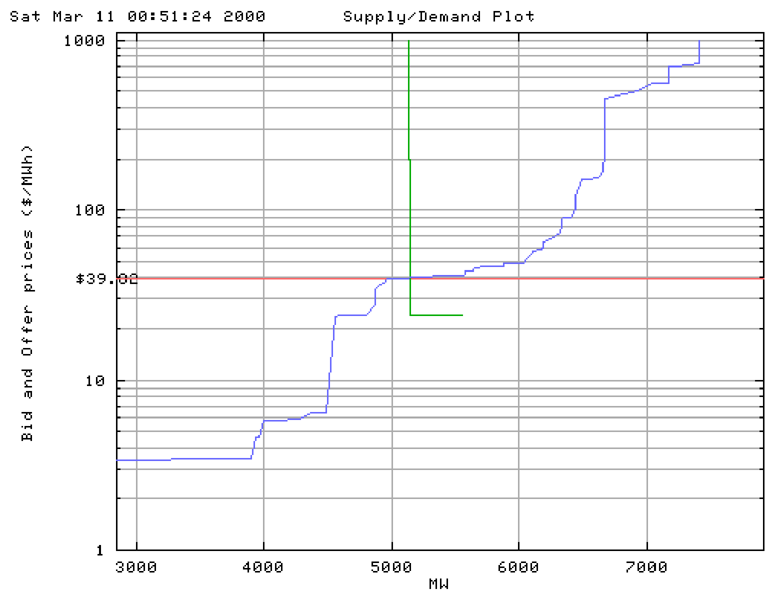

Figure 1: The Alberta Pool Merit Order graph, late March 11, 2000. Source: Power Pool of Alberta [4].

A daily periodic factor in the price is modelled by

$$g_t := \exp\left[\beta \int_0^t \sin(2\pi s + \psi) ds\right]. \tag{2}$$

Suppose $\overline{Z} = \{\overline{Z}_t, t \geq 0\}$ represents the number of large generating plants on line (large generation capacity) at time t. Here, $\overline{Z}_t$ takes values in the set $\{0, 1, \ldots, N-1\}$ and we suppose $\overline{Z}$ evolves as a homogeneous Markov chain. Without loss of generality we can consider the related Markov chain $\mathbf{Z} = \{\mathbf{Z}_t, t \geq 0\}$ whose state space is the set of unit vectors

$$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}, \quad \mathbf{e}_i = (0, \ldots, 1, \ldots, 0)' \in \mathbf{R}^N.$$

If $\mathbf{N}$ is the vector $(0, 1, \ldots, N-1)' \in \mathbf{R}^N$ then $\overline{Z}$ is the inner product

$$\overline{Z}_t = \langle \mathbf{Z}_t, \mathbf{N} \rangle. \tag{3}$$

Suppose $\mathbf{A}$ is the transition rate matrix of $\mathbf{Z}$ (or $\overline{Z}$). That is, with $p_t^i = p(\mathbf{Z}_t = \mathbf{e}_i)$ and

$$\mathbf{p}_t = \left(p_t^1, p_t^2, \ldots p_t^N\right)' \in \mathbf{R}^N$$

$$\frac{d\mathbf{p}_t}{dt} = \mathbf{A}\mathbf{p}_t.$$

As in Elliott, Aggoun and Moore [2], it is easily seen that

$$\mathbf{Z}_t = \mathbf{Z}_0 + \int_0^t \mathbf{A}\mathbf{Z}_s \, ds + \mathbf{M}_t \tag{4}$$

where $\mathbf{M} = \{\mathbf{M}_t, t \geq 0\}$

is a vector martingale with respect to the filtration generated by $\mathbf{Z}$.

Suppose $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots \mu_N)^{'} \in \mathbf{R}^N$ is a vector of mean-reversion levels for each state of large generation capacity.. We further suppose there is a scalar diffusion factor $X = \{X_t, t \geq 0\}$ modelling the logarithm of the deseasonalized price process. X has dynamics given by

$$dX_t = -\kappa \left(X_t - \langle \boldsymbol{\mu}, \mathbf{Z}_t \rangle \right) dt + \sigma d\omega_t \ . \tag{5}$$

Here $\omega = \{\omega_t, t \geq 0\}$ is a standard Brownnian motion on $(\Omega, \mathcal{F}, P)$, so X_t follows a mean-reverting process with long-run mean μ and strength of mean reversion κ. The half life of a deviation from the long-run mean is $\log 2/\kappa$.

When there are k large generators on line we suppose the spot price is weighted by a factor α_k. Write $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{N-1})' \in \mathbf{R}^N$. The spot price of electricity is then modelled as

$$S_t = f_t g_t \exp X_t \cdot \langle \boldsymbol{\alpha}, \mathbf{Z}_t \rangle \, . \tag{6}$$

The vector $\boldsymbol{\alpha}$ allows for variation in the spot price level according to the number of large generators on line, but in the special case $\boldsymbol{\alpha} = (1, 1, \ldots, 1)' \in \mathbb{R}^N$ there is no such variation.

3 Calibration

The calibration of the Markov chain $\mathbf{Z}$ is first discussed. Clearly $\overline{Z}$, (and therefore, $\mathbf{Z}$) is observed and known.

The jump in the price when the number of generators $\overline{Z}_t$ jumps is also observed, so the vector $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{k-1})'$ can be estimated.

Adapting arguments in [3] we can also see that the matrix $\mathbf{A} = [a_{ji}]$ matrix is estimated as

$$a_{ji} = \begin{cases} n_t^{ij}/m_t^i, & \text{if } i \neq j \\ -\sum_{k \neq i, k=1,\ldots,N} a_{ki} & \text{if } i = j, \end{cases} \tag{7}$$

where

$$n_t^{ij} = \text{the number of jumps of } \mathbf{Z} \text{ from state } i \text{ to } j \text{ up to time } t$$
$$m_t^{i} = \text{the length of time } \mathbf{Z} \text{ has spent in state } i \text{ up to time } t$$

The sample Markov probability transition matrix up to time t is

$$\mathbf{\Pi} := \mathbf{I} + \mathbf{A} \tag{8}$$

The parameters γ, ϕ, β, ψ, σ, κ, and $\boldsymbol{\mu}$ must then be estimated. We can use the EM (Expectation Maximization) algorithm in a filtered form.

Suppose the spot price S is observed over time periods of length h. Then

$$\begin{aligned}\log \frac{S_{t+h}}{S_t} = \gamma \int_t^{t+h} \sin\left(\frac{2\pi}{365}s + \phi\right) ds \\ + \beta \int_t^{t+h} \sin(2\pi s + \psi) ds - \kappa \int_t^{t+h} (X_s - \langle \boldsymbol{\mu}, \mathbf{Z}_t \rangle)\, ds \\ + \sigma(\omega_{t+h} - \omega_t) + \log\langle \boldsymbol{\alpha}, \mathbf{Z}_{t+h}\rangle - \log\langle \boldsymbol{\alpha}, \mathbf{Z}_t\rangle \ .\end{aligned} \tag{9}$$

We observe S and Z, and have already estimated $\boldsymbol{\alpha}$ so

$$Y_t := \log S_{t+h} - \log S_t$$

is known, as is

$$\delta_t := \log\langle \boldsymbol{\alpha}, \mathbf{Z}_{t+h}\rangle - \log\langle \boldsymbol{\alpha}, \mathbf{Z}_t\rangle \ .$$

Consequently we have a sequence of observations $Y = \{Y_t, t \geq 0\}$ with the approximate discrete distribution

$$Y_t \simeq \gamma \sin\left(\frac{2\pi}{365}t + \phi\right) h + \beta \sin(2\pi t + \phi) h - \kappa(X_t - \langle \boldsymbol{\mu}, \mathbf{Z}_t\rangle) h + \sigma\sqrt{h} B_t + \delta_t. \tag{10}$$

In this expression $B = \{B_t, t \geq 0\}$ is a sequence of i.i.d. $N(0,1)$ random variables on $(\Omega, \mathcal{F}, P)$.

We shall consider observation times at the partition points $\{nh : n \in \mathbb{Z}^+\}$ and, with abuse of notation, suppose t takes non-negative integer values and that $t \in \{nh;\, n \in \mathbb{Z}^+\}$.

Write $\phi(y) = \dfrac{1}{2\pi} \exp\left(-\dfrac{y^2}{2}\right)$ for the $N(0,1)$ density. Then the likelihood ratio for this model, up to time T, is

$$\Lambda_T := \prod_{t=1}^{T} \lambda_t \,. \tag{11}$$

where

$$\lambda_t = \frac{\phi\left(\frac{Y_t - \gamma\sin\left(\frac{2\pi}{365}t+\phi\right)h - \beta\sin(2\pi t+\psi)h + \kappa(X_t - \langle \boldsymbol{\mu}, \mathbf{Z}_t\rangle)h - \delta_t}{\sigma\sqrt{h}}\right)}{\sigma\sqrt{h}\,\phi(Y_t)} \tag{12}$$

Therefore,

$$\log\Lambda_T = -T\log(\sigma\sqrt{h}) - \frac{1}{2\sigma^2 h}\sum_{t=1}^{T}\left(Y_t - \gamma\sin\left(\frac{2\pi}{365}t+\phi\right)h \right.$$
$$\left. + \beta\sin(2\pi t+\psi)h + \kappa(X_t - \langle \boldsymbol{\mu}, \mathbf{Z}_t\rangle)h - \delta_t\right)^2 - \sum_{t=1}^{T}\frac{Y_t^2}{2}\,. \tag{13}$$

We observe the Y_t and the δ_t. We estimate γ, ϕ, β, ψ, σ, κ and $\boldsymbol{\mu}$ recursively using filtering and the EM algorithm. (Note we do not observe the X_t in the above sum.)

The EM algorithm starts with some prior values γ_0, ψ_0, β_0, ϕ_0, σ_0, κ_0, $\boldsymbol{\mu}_0$ and re-estimates each of these parameters, one at a time, recursively, given the data.

Suppose, given $Y_1, \ldots, Y_T$ and $\delta_1, \ldots, \delta_T$, we have, after p iterations, estimates

$$\gamma_p,\ \phi_p,\ \beta_p,\ \psi_p,\ \sigma_p,\ \kappa_p,\ \boldsymbol{\mu}_p\ .$$

Write $\mathcal{Y}_t = \sigma\{Y_0, Y_1, \ldots, Y_t\}$ for the filtration generated by Y.

Suppose, initially, we wish to estimate γ_{p+1}. We consider

$$E_p[\log\Lambda_T \mid \mathcal{Y}_T] = Q_T(\gamma_p, \phi_p, \beta_p, \ldots)$$

From the first order conditions we consider $\dfrac{\partial Q_T}{\partial \gamma_p} = 0$ and the value of γ_p which satisfies this.

This gives $\hat{\gamma}_{p+1}$. We proceed similarly to update the other coefficients.

For example,

$$
\begin{aligned}
Q_T =& E[\log \Lambda_T \mid \mathcal{Y}_T] \\
=& -T\log\sigma - \frac{T}{2}\log h \\
& - \frac{1}{\sigma^2 h} E\left[\sum_{t=1}^{T}\left(Y_t - \gamma_p \sin\left(\frac{2\pi}{365}t + \phi_p\right)h - \beta_p \sin(2\pi t + \psi_p)h\right.\right. \\
& \left.\left. + \kappa_p(X_t - \langle \boldsymbol{\mu}_p, \mathbf{Z}_t\rangle)h - \delta_t\right)^2 \mid \mathcal{Y}_t\right] - \sum_{t=1}^{T}\frac{Y_t^2}{2} .
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{\partial Q_T}{\partial \gamma_p} =& \frac{2}{\sigma^2} E\left[\sum_{t=1}^{T}\sin\left(\frac{2\pi}{365}t + \phi_p\right)\left(Y_t - \gamma_p \sin\left(\frac{2\pi}{365}t + \phi_p\right)h\right.\right. \\
& \left.\left. - \beta_p \sin(2\pi t + \psi_p)h + \kappa_p(X_t - \langle \boldsymbol{\mu}_p, \mathbf{Z}_t\rangle)h - \delta_t\right) \mid \mathcal{Y}_T\right] .
\end{aligned}
$$

Putting $\frac{\partial Q_T}{\partial \gamma_p} = 0$ we have

$$
\begin{aligned}
\hat{\gamma}_{p+1} =& \frac{1}{h\sum_{t=1}^{T}\sin\left(\frac{2\pi}{365}t + \phi_p\right)^2}\sum_{t=1}^{T}\sin\left(\frac{2\pi}{365}t + \phi_p\right) \\
& \times \left(Y_t - \beta_p \sin(2\pi t + \psi_p)h - \delta_t - \kappa_p \langle \boldsymbol{\mu}_p, \mathbf{Z}_t\rangle hT + \kappa_p E\left[\sum_{t=1}^{T} X_t \mid \mathcal{Y}_T\right] h\right)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\frac{\partial Q_T}{\partial \sigma_p} =& -\frac{T}{\sigma_p} \\
& + \frac{1}{\sigma_p^3 h} E\left[\sum_{t=1}^{T}\left(Y_t - \gamma_p \sin\left(\frac{2\pi}{365}t + \phi_p\right)h - \beta_p \sin(2\pi t + \psi_p)h\right.\right. \\
& \left.\left. \kappa_p(X_t - \langle \boldsymbol{\mu}_p, \mathbf{Z}_t\rangle)h - \delta_t\right)^2 \mid \mathcal{Y}_T\right]
\end{aligned}
$$

giving an estimate

$$\hat{\sigma}^2_{p+1} = \frac{1}{T} E\left[\sum_{t=1}^{T}\left(Y_t - \gamma_p \sin\left(\frac{2\pi}{365}t + \phi_p\right)h - \beta_p \sin(2\pi t + \psi_p)h \right.\right.$$
$$\left.\left. + \kappa_p(X_t - \langle \boldsymbol{\mu}_p, \mathbf{Z}_t\rangle)h - \delta_t^2\right)^2 \mid \mathcal{Y}_T\right]$$

In the sum on the right Y_t, δ_t, $t = 1, \ldots, T$, are known, as are the estimates γ_p, ϕ_p, β_p, ψ_p, κ_p, $\boldsymbol{\mu_p}$.

We are faced with estimating sums of the form $E\left[\sum_{t=1}^{T} F_1(t, Y_t, \delta_t)X_t \mid \mathcal{Y}_T\right]$ and $E\left[\sum_{t=1}^{T} G_1(t, Y_t, \delta_t)X_t^2 \mid \mathcal{Y}_T\right]$.

These can be recursively using the extensions to the Kalman filter given in [3].

Similarly

$$\frac{\partial Q_T}{\partial \phi_p} = -\frac{1}{\sigma^2} E\left[\sum_{t=1}^{T} \cos\left(\frac{2\pi}{365}t + \phi_p\right)\left(Y_t - \gamma_p \sin\left(\frac{2\pi}{365}t + \phi_p\right)h \right.\right.$$
$$\left.\left. - \beta_p \sin(2\pi t + \psi_p)h - \delta_t\right) \mid \mathcal{Y}_T\right] \tag{14}$$

This again involves $E\left[\sum_{t=1}^{T} X_t \mid \mathcal{Y}_T\right]$.

We look for the implicit solution $\hat{\phi}_{p+1}$ to $\frac{\partial Q_T}{\partial \phi_p} = 0$. Similarly

$$\frac{\partial Q_T}{\partial \psi_p} = -\frac{1}{\sigma^2} E\left[\sum_{t=1}^{T} \cos(2\pi t + \psi)\left(Y_t - \gamma_p \sin\left(\frac{2\pi t}{365} + \phi_p\right)h \right.\right.$$
$$\left.\left. - \beta_p \sin(2\pi t + \psi)h + \kappa_p(X_t - \langle \boldsymbol{\mu}_p, \mathbf{Z}_t\rangle)h - \delta_t\right) \mid \mathcal{Y}_T\right].$$

We set $\frac{\partial Q_T}{\partial \psi_p} = 0$ and look for the implicit solution $\hat{\psi}_{p+1}$. $\frac{\partial Q_T}{\partial \beta_p}$ is similar to

$\frac{\partial Q_T}{\partial \gamma_p}$. For κ_p we have

$$\frac{\partial Q_T}{\partial \kappa_p} = -\frac{1}{\sigma^2} E\left[\sum_{t=1}^{T}\left(Y_t - \gamma_p \sin\left(\frac{2\pi}{365}t + \phi_p\right) - \beta_p \sin(2\pi t + \psi_p) - \delta_t \right.\right.$$
$$\left.\left. - \kappa_p(X_t - \langle \boldsymbol{\mu}_p, \mathbf{Z}_t\rangle)h\right)(X_t - \langle \boldsymbol{\mu}_p, \mathbf{Z}_t\rangle) \mid \mathcal{Y}_T\right]$$

Setting $\frac{\partial Q_T}{\partial \kappa_p} = 0$ gives

$$\hat{\kappa}_{p+1} = \frac{1}{E\left[\sum_{t=1}^{T}(X_t - \langle \boldsymbol{\mu}_p, \mathbf{Z}_t\rangle)^2 \mid \mathcal{Y}_T\right]} E\left[\sum_{t=1}^{T}\left(Y_t - \gamma_p \sin\left(\frac{2\pi t}{365} + \phi_p\right)\right.\right.$$
$$\left.\left. \beta_p \sin(2\pi t + \psi_p) - \delta_t\right)(X_t - \langle \boldsymbol{\mu}_p, \mathbf{Z}_t\rangle) \mid \mathcal{Y}_T\right] .$$

Again, the estimate involves things of the form $E\left[\sum_{t=1}^{T} F_2(t, Y_t, \delta_t)X_t^2 \mid \mathcal{Y}_T\right]$ and $E\left[\sum_{t=1}^{T} G_2(t, Y_t, \delta_t)X_t \mid \mathcal{Y}_T\right]$. Recursive estimates for those can be found as in the paper [3].

We can take a similar approach to estimate the components of the vector $\boldsymbol{\mu}$. The derivatives are vectors.

$$\frac{\partial Q_T}{\partial \boldsymbol{\mu}_p} = \frac{2\kappa_p}{\sigma^2} E\left[\sum_{t=1}^{T}\left(Y_t - \gamma_p \sin\left(\frac{2\pi}{365}t + \phi_p\right) h - \beta_p \sin(2\pi t + \psi_p)h \right.\right.$$
$$\left.\left. + \kappa_p(X_t - \langle \boldsymbol{\mu}_p, \mathbf{Z}_t\rangle)\mathbf{Z}_t h - \delta_t\right) \mid \mathcal{Y}_T\right]$$

Recall that the vector $\mathbf{Z}_t$ is zero in all components except one and is one in the component corresponding to the generation state at time t . Thus, at time t, if $\mathbf{Z}_t = \mathbf{e}_j$,we can only update the component $\mu_{p,j}$ of $\boldsymbol{\mu}_p$ corresponding to the longrun mean for the generation state j. Define $T(j) = \sum_{\{t \mid \mathbf{Z}_t = \mathbf{e}_j\}} 1$

to the be total time spent visiting state j. Setting $\dfrac{\partial Q_T}{\partial \mu_{p,j}} = 0$ gives

$$\hat{\mu}_{p+1,j} = \frac{1}{\kappa_p h T(j)} \sum_{\{t|\mathbf{Z}_t = \mathbf{e}_j\}} \left(Y_t - \gamma_p \sin\left(\frac{2\pi t}{365} + \phi_p\right) h \right.$$
$$\left. - \beta_p \sin(2\pi t + \psi_p) h - \delta_t + \kappa_p h E[X_t \mid \mathcal{Y}_T] \right) .$$

4 European Calls

We have taken the model for the spot price to be S where

$$S_t = f_t g_t \exp X_t \cdot \langle \boldsymbol{\alpha}, \mathbf{Z}_t \rangle$$

Note that for $0 \leq t \leq T < \infty$

$$X_T = e^{-\kappa(T-t)} \left[X_t + \kappa \int_t^T \langle \boldsymbol{\mu}, \mathbf{Z}_s \rangle e^{\kappa(s-t)}\, ds + \sigma \int_t^T e^{\kappa(s-t)}\, dW_s \right].$$

Write $\left\{\mathcal{F}_t^X\right\}$ (resp $\left\{\mathcal{F}_t^{\mathbf{Z}}\right\}$) for the complete, right continuous filtrations generated by X and $\mathbf{Z}$, respectively. If we know a full history of $\mathbf{Z}$ to time T, we know information in $\mathcal{F}_T^{\mathbf{Z}}$.

Now

$$\begin{aligned} E\left[X_T | \mathcal{F}_T^{\mathbf{Z}} \vee X_t\right] &= \exp\left(-\kappa\left(T-t\right)\right) \left[X_t + \kappa \int_t^T \langle \boldsymbol{\mu}, \mathbf{Z}_s \rangle e^{\kappa(s-t)} ds \right] \\ &= \mu_{t,T}\left(\mathbf{Z}\right), \text{ say.} \end{aligned}$$

Also,

$$\begin{aligned} \text{Var}\left[X_T | X_t \vee \mathcal{F}_T^{\mathbf{Z}}\right] &= \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa(T-t)}\right) \\ &= \nu_{t,T}^2 \text{ , say.} \end{aligned}$$

Given $\mathcal{F}_T^{\mathbf{Z}}$, that is, given a particular history of Z up to time T, and given the parameters of our model, S_T is log-normal.

Therefore,

$$F\left(t, T, Z\right) := E\left[S_T | \mathcal{F}_t^X \vee \mathcal{F}_T^{\mathbf{Z}}\right]$$

$$= E\left[f_T g_T e^{X_T}\langle\boldsymbol{\alpha},\mathbf{Z}_T\rangle|\mathcal{F}_t^X\vee\mathcal{F}_T^{\mathbf{Z}}\right]$$
$$= f_T g_T\ \exp\left(\mu_{t,T}(\mathbf{Z})+\frac{1}{2}\,\nu_{t,T}^2\right)\langle\boldsymbol{\alpha},\mathbf{Z}_t\rangle.$$

This is a conditional futures price in our model, given knowledge of $\mathcal{F}_T^{\mathbf{Z}}$; we assume it is also a conditional forward price, (given $\mathcal{F}_T^{\mathbf{Z}}$).

Now $F(t,T,\mathbf{Z})$ is a conditional forward price, knowing the behaviour of Z_u for $0\le u\le T$. To find a true forward price we must condition $F(t,T,\mathbf{Z})$ on the history of X and $\mathbf{Z}$ only to time t, that is, knowing $\mathcal{F}_t^X\vee\mathcal{F}_T^{\mathbf{Z}}$.

The true forward price $F(t,T)$ is then

$$E\left[F(t,T,Z)\,|\,\mathcal{F}_t^X\vee\mathcal{F}_t^{\mathbf{Z}}\right] \tag{15}$$

To determine this quantity, we need the following lemma:

Lemma 1
Write

$$\mathbf{D}_u=\kappa e^{\kappa(u-T)}\begin{pmatrix}\mu_1 & 0 & \cdots & 0\\ 0 & \mu_2 & & \\ \vdots & & \ddots & \vdots\\ 0 & & \cdots & \mu_N\end{pmatrix}$$

and suppose $\boldsymbol{\Phi}(t,u)$ *is the matrix solution of*

$$\frac{d\boldsymbol{\Phi}(t,u)}{du}=(\mathbf{D}_u+\mathbf{A})\,\boldsymbol{\Phi}(t,u)$$
$$\boldsymbol{\Phi}(t,t)=\mathbf{I}$$

Then

$$E\left[\exp\left(\kappa\int_t^T e^{-\kappa(T-u)}\langle\boldsymbol{\mu},\mathbf{Z}_u\rangle du)\,|\,\mathcal{F}_t^X\vee\mathcal{F}_T^{\mathbf{Z}}\right)\right]=\langle\boldsymbol{\Phi}(t,T)\mathbf{Z}_t,\mathbf{1}\rangle$$

where

$$\mathbf{1}=(1,1,\dots,1)'\in\mathbb{R}^N$$

Proof:
Define the scalar

$$\Gamma_{t,u}=\exp\left[\kappa\int_t^u e^{-\kappa(T-s)}\langle\boldsymbol{\mu},\mathbf{Z}_s\rangle\,ds\right]$$

and set

$$\mathbf{Y}_u = \Gamma_{t,u}\mathbf{Z}_u \in \mathbb{R}^N$$

Then, using equation 4 to expand dZ_u, we have:

$$\begin{aligned} d\mathbf{Y}_u &= \kappa e^{\kappa(u-T)}\langle \boldsymbol{\mu}, \mathbf{Z}_u\rangle \Gamma_{t,u}\,\mathbf{Z}_u du + \mathbf{A}\,\Gamma_{t,u}\,\mathbf{Z}_u\,du + \Gamma_{t,u}\,d\mathbf{M}_u \\ &= \kappa e^{\kappa(u-T)}\langle \boldsymbol{\mu}, \mathbf{Z}_u\rangle \mathbf{Y}_u\,du + \mathbf{A}\mathbf{Y}_u\,du + \Gamma_{t,u}\,d\mathbf{M}_u \end{aligned}$$

Therefore,

$$\begin{aligned} E[\mathbf{Y}_t \,|\, \mathcal{F}_t^{\mathbf{Z}}] &= E[\mathbf{Y}_t \,|\, \mathbf{Z}_t] \\ &= \mathbf{Z}_t + \int_t^T (\mathbf{D}_u + \mathbf{A}) E[\mathbf{Y}_u \,|\, \mathbf{Z}_t]\,du \end{aligned} \tag{16}$$

and so

$$E[\mathbf{Y}_t \,|\, \mathbf{Z}_t] = \boldsymbol{\Phi}(t,T)\mathbf{Z}_t \tag{17}$$

Then

$$E\left[\exp\left(\kappa \int_t^T e^{-\kappa(T-s)}\langle \boldsymbol{\mu}, \mathbf{Z}_s\rangle\right) ds\right] = \langle \boldsymbol{\Phi}(t,t)\mathbf{Z}_t, \mathbf{1}\rangle$$

QED

Corollary 1
The true forward price at time t is then

$$\begin{aligned} F(t,T) &= E\left[S_T \,|\, \mathcal{F}_t^X \vee \mathcal{F}_t^{\mathbf{Z}}\right] \\ &= E\left[F(t,T,\mathbf{Z}) \,|\, \mathcal{F}_t^{\mathbf{Z}}\right] \\ &= E\left[F(t,T,\mathbf{Z}) \,|\, \mathbf{Z}_t\right] \\ &= f_T g_T \exp\left(e^{-\kappa(T-t)}X_t + \frac{\nu_{t,T}^2}{2}\right)\langle \boldsymbol{\Phi}(t,t)\mathbf{Z}_t, \boldsymbol{\alpha}\rangle \end{aligned}$$

We now wish to consider the time-0 value of a call option (exercised at time t) on this forward price (with a forward maturity date T), where $0 \le t \le T$. That is, we wish to evaluate

$$C(0,t,T,X_0,\mathbf{Z}_0) = E\left[e^{-rt}(F(t,T) - K)^+ \,|\, X_0, \mathbf{Z}_0\right]$$

Now

$$F(t,T) = f_T g_T \exp\left(e^{-\kappa(T-t)}\left(X_0 + \kappa\int_0^t \langle \boldsymbol{\mu}, \mathbf{Z}_s\rangle e^{\kappa s}\,ds + \sigma\int_0^t e^{\kappa s}\,dW_s\right) + \frac{\nu_{t,T}^2}{2}\right)$$
$$\times \langle \boldsymbol{\Phi}(t,t)\mathbf{Z}_t, \boldsymbol{\alpha}\rangle$$
$$= G(t,T,\mathbf{Z})\exp\left(e^{-\kappa T}\sigma\int_0^t e^{\kappa s}\,dW_s - \frac{\sigma^2(0,t)}{2}\right)$$

where

$$\sigma^2(0,t) = \frac{\sigma^2}{2\kappa}\left(e^{-2\kappa(T-t)} - e^{-2\kappa T}\right)$$

and

$$G(t,T,\mathbf{Z}) = f_T g_T \exp\left(e^{-\kappa T}\left(X_0 + \kappa\int_0^t \langle \boldsymbol{\mu}, \mathbf{Z}_s\rangle e^{\kappa s}\,ds + \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa T})\right)\right)$$
$$\times \langle \boldsymbol{\Phi}(t,t)\mathbf{Z}_t, \boldsymbol{\alpha}\rangle$$

Again, suppose we know the history of $\mathbf{Z}_u$ $(0 \le u \le t)$, that is, given $\mathcal{F}_t^{\mathbf{Z}}$, consider

$$C(0,t,T,X_0,\mathbf{Z}) := E\left[e^{-rt}(F(t,T) - K)^+ \mid X_0, \mathcal{F}_t^{\mathbf{Z}}\right] \tag{18}$$

Given $\mathcal{F}_t^{\mathbf{Z}}$, $F(t,T)$ is lognormal so with

$$d(t,T) = \frac{\ln\frac{G(t,T,\mathbf{Z})}{\sigma(0,t)}}{\sigma(0,t)} + \frac{\sigma(0,t)}{2}$$

we have

$$C(0,t,T,X_0,\mathbf{Z}) = e^{-rt}\left(G(t,T,\mathbf{Z})N(d(t,T)) - KN(d(t,T) - \sigma(0,t))\right)$$
$$= W(\Xi, \mathbf{Z}_t), \text{ say,} \tag{19}$$

where

$$\Xi = \kappa\int_0^t \langle \boldsymbol{\mu}, \mathbf{Z}_s\rangle e^{\kappa s}\,ds \tag{20}$$

To determine the actual call option value, we must finally condition out $(\Xi, \mathbf{Z}_t)$ given $\mathbf{Z}_0$. We do this in Fourier transform space, using the characteristic function of Ξ.

Suppose $\lambda(\xi, \mathbf{e}_j)$ is the density function of Ξ and the condition[2] $\mathbf{Z}_t = \mathbf{e}_j$, so that for any integrable function $G(\Xi)$

$$E\left[G(\Xi)\langle \mathbf{Z}_t, \mathbf{e}_j\rangle\right] = \int_{-\infty}^{\infty} G(\xi)\lambda(\xi, \mathbf{e}_j)\, d\xi \; .$$

Then the Fourier transform (characteristic function) of $\lambda(\xi, \mathbf{e}_j)$ is

$$\begin{aligned}\hat{\lambda}(u, \mathbf{e}_j) &= E\left[\frac{\exp(iu\Xi)}{\sqrt{2\pi}}\langle \mathbf{Z}_t, \mathbf{e}_j\rangle\right] \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iu\xi}\,\lambda(\xi, \mathbf{e}_j)\, d\xi\end{aligned} \tag{21}$$

Lemma 2
The characteristic functions are computed as

$$\hat{\lambda}(u, \mathbf{e}_j) = \langle \mathbf{\Psi}\,(t)\,\mathbf{Z}_0, \mathbf{e}_j\rangle$$

where the matrix $\mathbf{\Psi}$ *is the unique solution of*

$$\frac{d\mathbf{\Psi}\,(t)}{dt} = \left(\mathbf{D}_t^{'} + \mathbf{A}\right)\mathbf{\Psi}\,(0, t)\,,$$

with

$$\begin{aligned}\mathbf{\Psi}\,(0) &= \mathbf{I},\;\; and \\ \mathbf{D}_t^{'} &= iue^{\kappa t}\cdot diag\,(\mu_{1,}\mu_{2,}..., \mu_N)\,.\end{aligned}$$

Proof:
The proof is very similar to that of Lemma 1. Consider the vector process

[2] Strictly speaking, λ is not a conditional density function because it does not integrate to 1. Instead, it integrates to the probability of the event defined by $\mathbf{Z}_t = \mathbf{e}_j$. To make this a conditional probability, we can divide by the probability of this event, but it only makes the notation more cumbersome and creates concerns that this event may have probability zero. Thus, strictly speaking we only have a Fourier transform and not a characteristic function. However, the properties that we use are those of Fourier transforms.

$$\mathbf{Y}_t := \mathbf{Z}_t \exp\left(iu\kappa \int_0^t \langle \boldsymbol{\mu}, \mathbf{Z}_s \rangle e^{\kappa s} ds \right)$$

Recall equation (4), that $d\mathbf{Z}_t = \mathbf{A}\mathbf{Z}_t dt + d\mathbf{M}_t$. Then

$$\begin{aligned} d\mathbf{Y}_t &= \left(iu\kappa \langle \boldsymbol{\mu}, \mathbf{Z}_t \rangle e^{\kappa t} \mathbf{Y}_t \right) dt + \mathbf{A}\mathbf{Y}_t \, dt + \mathbf{Y}_{t-} \, d\mathbf{M}_t \\ &= \left(\mathbf{D}_t^{'} + \mathbf{A} \right) \mathbf{Y}_t \, dt + \mathbf{Y}_{t-} \, d\mathbf{M}_t \end{aligned}$$

where

$$\begin{aligned} \mathbf{D}_t^{'} &= iu\kappa e^{\kappa t} \begin{pmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & \mu_N \end{pmatrix} \\ &= iu\kappa e^{\kappa t} \cdot \operatorname{diag} \boldsymbol{\mu}. \end{aligned}$$

As $\mathbf{M}$ is a martingale we have

$$dE\left[\mathbf{Y}_t\right] = \left(\mathbf{D}_t^{'} + \mathbf{A} \right) E\left[\mathbf{Y}_t\right] \, dt$$

so

$$E\left[\mathbf{Y}_t\right] = \boldsymbol{\Psi}(t) \, \mathbf{Z}_0$$

where $\boldsymbol{\Psi}(t)$ is the fundamental matrix solution of

$$\begin{aligned} \frac{d\boldsymbol{\Psi}(t)}{dt} &= \left(\mathbf{D}_t^{'} + \mathbf{A} \right) \boldsymbol{\Psi}(t), \\ \boldsymbol{\Psi}(0) &= \mathbf{I}. \end{aligned}$$

Noting

$$\begin{aligned} \mathrm{E}\left[\mathbf{Y}_t\right] &= \mathrm{E}\left[\mathbf{Z}_t \exp\left(iu\kappa \int_0^t \langle \boldsymbol{\mu}, \mathbf{Z}_s \rangle e^{\kappa s} ds \right) \right] \\ &= \mathrm{E}\left[\mathbf{Z}_t \sum_{j=1}^{N} \langle \mathbf{Z}_t, \mathbf{e}_j \rangle \exp\left(iu\kappa \int_0^t \langle \boldsymbol{\mu}, \mathbf{Z}_s \rangle e^{\kappa s} ds \right) \right] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{N} \mathrm{E}\left[\mathbf{e}_j \langle \mathbf{Z}_t, \mathbf{e}_j \rangle \exp\left(iu\kappa \int_0^t \langle \boldsymbol{\mu}, \mathbf{Z}_s \rangle e^{\kappa s} ds \right)\right] \\
&= \sum_{j=1}^{N} \mathbf{e}_j E\left[\frac{\exp(iu\Xi)}{\sqrt{2\pi}} \langle \mathbf{Z}_t, \mathbf{e}_j \rangle \right] \\
&= \sum_{j=1}^{N} \mathbf{e}_j \hat{\lambda}\left(u, \mathbf{e}_j\right) \ .
\end{aligned}
$$

The second equality comes from observing that the state vector $\mathbf{Z}_t$ equals exactly one of the vectors $\mathbf{e}_j$ and has an inner product of 1 with that vector and zero with all the others. The third equality comes from reversing the summation and the expectation and again noting that the inner product $\langle \mathbf{Z}_t, \mathbf{e}_j \rangle$ equals 1 if and only if $\mathbf{Z}_t = \mathbf{e}_j$. The fourth equality comes from the definition of Ξ in equation (20). The last equality comes from the definition (21).

This establishes the result.

QED

We now wish to determine

$$
\begin{aligned}
E\left[W(\Xi, \mathbf{Z}_t)\right] &= \sum_{j=1}^{N} E\left[W(\Xi, \mathbf{e}_j) \langle \mathbf{Z}_t, \mathbf{e}_j \rangle\right] \\
&= \sum_{j=1}^{N} \int_{-\infty}^{\infty} W(\xi, \mathbf{e}_j) \lambda(\xi, \mathbf{e}_j) \, d\xi
\end{aligned}
$$

By Plancherel's theorem (see, e.g. Yosida [6]) this equals

$$
\sum_{j=1}^{N} \int_{-\infty}^{\infty} \hat{W}(u, \mathbf{e}_j) \hat{\lambda}(u, \mathbf{e}_j) \, du
$$

where

$$
\hat{W}(u, \mathbf{e}_j) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iu\xi} W(\xi, \mathbf{e}_j) \, d\xi
$$

5 Empirical Estimation

The Power Pool of Alberta publishes hourly data on generator availability, demand, supply and pool price on the Internet, but not in archived form.

We were fortunate[3] to receive the Power Pool's own hourly data set of pool prices, demand and supply by each of the generators in Alberta for the period January 1, 1996 to March 31, 2000. Here we present the analysis of the 4-year data set for 1996-1999.

5.1 Estimating the Markov Process of On-line Generators

An examination of the data revealed that the Power Pool had 14 large coal-fired generators with a capacity in excess of 250 megawatts. All the other generators were much smaller. While the generators don't always develop power at their full capacity, they can't be run for significant periods of time at much less than a load of 90%, for at lower loads they become unstable. Thus, it was not difficult to distinguish the periods when a large generator was on-line from those when it was off-line. We chose the cut-off of 10 megawatts of power being generated by these units. We identified periods of time when as few as 9 generators were running and periods when as many as 14 generators were running. Table 1 provides summary statistics of the length of time spent in each of these states and the time to wait before transition.

Table 2 is the Markov transition matrix from the state of identified by the top of each column to the state identified by the first item of each row.

Table 3 shows the Markov transitions after convolution for 1 day, 2 days, ..., 90 days. Note that the columns of the 30 and 90 day matrices are essentially identical, and this vector is essentially the ergodic probability vector that represents the long-term steady-state probability of being in each of the states. The most common state is that with 13 generators running (35.2% probability). For shorter periods of time, the transitions are most commonly to increase or decrease by only one generator.

5.2 Estimating Price Seasonality

The Expectation Maximization algorithm requires an initial estimate of parameters. We chose to estimate these parameters in a two step process in

[3]We gratefully acknowledge the support of Maury Parsons and Owen Craig in obtaining the data.

Table 1: Hourly status of numbers of large generators on-line 1996-99. To interpret this table, note that the first row indicates that there were 5 transitions from the state with 9 generators to the state with 10 generators. The mean time before the transition is 12.8 hours and the median is 10 hours. The minimum number of hours before the transition is 2 and the maximum is 29.

$\overline{Z}_t$	$\overline{Z}_{t+1}$	Visits	Mean Stay	Median	Min Stay	Max Stay
9	10	5	12.8	10	2	29
10	9	5	18.2	7	1	59
10	11	44	16.5	9	1	158
10	12	1	3.0	3	3	3
11	10	43	35.3	20	1	171
11	12	123	25.0	17	1	127
11	13	1	118.0	118	118	118
12	10	2	41.0	41	38	44
12	11	122	45.2	26	1	363
12	13	162	28.4	17	1	189
13	11	1	133.0	133	133	133
13	12	161	47.8	25	1	310
13	14	129	34.7	24	1	158
14	13	129	53.0	33	1	308

Table 2: Hourly Markov probability transition matrix $\Pi = [\pi_{ji}]$ showing the probability of a transition from state i to state j.

State	9	10	11	12	13	14
9	0.9218	0.0061	0	0	0	0
10	0.0781	0.9389	0.0093	0	0	0
11	0	0.0525	0.9645	0.012	0	0
12	0	0.0024	0.0258	0.9719	0.0131	0
13	0	0	0.0002	0.0157	0.9763	0.0186
14	0	0	0	0	0.0104	0.9813
Total	1	1	1	1	1	1

Table 3: Markov transition probabilities after 1, 3, 7, 30 and 90 days. The columns represent the initial states of 9, 10, . . . 14 generators and the rows represent the ending states of 9, 10, . . . 14 generators.

1 day					
0.1684	0.0311	0.0045	0.0005	0	4.41E-06
0.3992	0.2951	0.0819	0.014	0.0016	0.0001
0.3288	0.4622	0.5172	0.1507	0.025	0.0037
0.0925	0.1825	0.3253	0.5838	0.1837	0.0413
0.0103	0.0269	0.0651	0.2215	0.6306	0.284
0.0005	0.0019	0.0058	0.0292	0.1588	0.6706
3 days					
0.0158	0.0092	0.0051	0.0021	0.0006	0.0002
0.1183	0.0865	0.0588	0.0293	0.0109	0.0041
0.3714	0.3348	0.2827	0.1774	0.0817	0.0372
0.3392	0.3661	0.3842	0.3722	0.2569	0.1577
0.1301	0.1662	0.2134	0.3105	0.4229	0.4045
0.025	0.0368	0.0555	0.1082	0.2266	0.3961
7 days					
0.0033	0.003	0.0027	0.0021	0.0015	0.0011
0.0387	0.0358	0.0327	0.0269	0.0201	0.0155
0.198	0.1873	0.1747	0.151	0.1207	0.0989
0.3424	0.3358	0.3272	0.3082	0.2791	0.2543
0.2944	0.3045	0.3162	0.338	0.3644	0.3809
0.1229	0.1333	0.1463	0.1735	0.214	0.249
30 days					
0.0018	0.0018	0.0018	0.0018	0.0018	0.0018
0.0233	0.0233	0.0233	0.0233	0.0233	0.0233
0.1342	0.1342	0.1342	0.1341	0.1341	0.1341
0.2906	0.2906	0.2906	0.2906	0.2906	0.2905
0.3519	0.3519	0.352	0.352	0.352	0.352
0.1979	0.198	0.198	0.198	0.1981	0.1981
90 days					
0.0018	0.0018	0.0018	0.0018	0.0018	0.0018
0.0233	0.0233	0.0233	0.0233	0.0233	0.0233
0.1341	0.1341	0.1341	0.1341	0.1341	0.1341
0.2906	0.2906	0.2906	0.2906	0.2906	0.2906
0.352	0.3521	0.3521	0.3521	0.352	0.352
0.1981	0.1981	0.1981	0.1981	0.1981	0.1981

Table 4: Estimates of Seasonal Pool Price Effects

Log price	Coeff.	Std. Err.	t	Prob > $\|t\|$	Exp(Coeff.)
Constant	2.39237	0.00736	325.2	0.00	10.94
Time (Hours)	0.00004	0.00000	144.1	0.00	1.000039
Weekend	-0.15846	0.00592	-26.8	0.00	0.85
Sine Daily	-0.33711	0.00450	-74.9	0.00	0.71
Cosine Daily	-0.30585	0.00678	-45.1	0.00	0.74
Night-time	0.00464	0.01077	0.4	0.67	1.0046
Sine Annual	-0.08279	0.00387	-21.4	0.00	0.92
Cosine Annual	0.11860	0.00427	27.8	0.00	1.13
R-squared	0.529				
Daily Amp.	0.455				1.58
Peak Hour	15.2				
Annual Amp.	0.145				1.16
Peak Month	10.86				

which we first take logarithms of the data and estimate the (daily and annual) seasonal characteristics f_t and g_t of the prices and then deflate the process to get de-seasonalized process X_t. Then, the de-seasonalized process is studied for its mean-reversion characteristics and to extract the properties of the residuals or diffusion terms.

The Pool Price data exhibited a clear uptrend in price levels, so we estimated a trend along with the seasonality. The demand for electricity is also affected by some characteristics that do not follow a purely sinusoidal form, such as the distinction between weekday and weekend (identifying industrial demand) and the distinction between daylight hours and nighttime hours (identifying street and home lighting demand). Introducing these dummy variables also allows for the possibility of replacing the sinusoidal seasonals with seasonal dummy variables.

In addition, the price level adjustment $\langle \boldsymbol{\alpha}, \mathbf{Z}_t \rangle$ that depends on the number of on-line generators can be augmented to allowing a completely different seasonal and dummy model for each level of on-line generation, since there are only 6 different values of Z_t in the data.

Table 4 shows the results of the first-stage seasonal regression of log price on seasonal, dummy and trend variables. The last column provides the exponential transformation of the coefficient, which can be interpreted as a factor affecting the price level. For example, the exponential of the constant is 10.94, which suggests that the typical pool price at the beginning of the time period (January 1996), a typical pool price was \$10.94 per megawatt-hour, or in household terms, 1.094 cents per kilowatt-hour. In this model, time is measured in hours, so at the end of December 1999, there were $24 \times 365 \times 4 = 35,040$ hours. The time-trend led to a factor of $\exp(35,040 \times .00004) = 3.89$, so that a typical price was $10.94 \times 3.89 = \$42.60$ per megawatt-hour. This is a very substantial rate of increase and may not continue into the future.

The weekend had an effect of reducing the price by a factor 0.85.

Entering the night hours[4] only increased the price by a factor of 0.46%, and this was insignificant. This suggests that the increased demand as street lights and house lights are turned on does not have a significant price effect. Of course, this variable is correlated with the time of day, and the price in the evening was typically reduced by the reduced industrial demand, so a proxy effect with errors in the variables may be muting the statistical significance of this factor.

The daily and annual seasonal factors were estimated by noting that the integrals in equations (1) and (2) become sines with a phase angle shift. Thus, for example, the daily seasonal in logged prices is $-.33711 \sin(t \times 2\pi/24) - 0.30585 \cos(t \times 2\pi/24)$. Recall the trigonometric identities

$$a \sin(x + \phi) = a \cos(\phi) \sin(x) + a \sin(\phi) \cos(x) \tag{22}$$

$$a^2 = (a \cos(\phi))^2 + (a \sin(\phi))^2. \tag{23}$$

Associating the coefficients of the daily sine and cosine in the regression with $a \cos(\phi)$ and $a \sin(\phi)$, respectively, we can solve for the amplitude a by taking a square root of the sums of the squares of the coefficients. This gives a daily amplitude of 0.455 for the logarithm and a factor of 1.58 for the price. Thus, the peak price of the day is typically 158% of the mean price for the day. To determine the peak hour, we can take the inverse sines and cosines of the regression coefficients divided by the amplitude. Some care must be taken to interpret these correctly, since the square root might have been the negative root. Moreover, equation (22) gives a phase angle relative to the sine, which has its maximum at $\pi/2$, rather than 0. Thus, the daily peak is at a time

[4]We used software [5] by the US Naval Observatory to calculate the darkness hours, including twilight, for both Edmonton and Calgary and assigned a dummy variable of 1 for darkness. We averaged the dummy variable for Edmonton and Calgary. We are grateful to Jonathan Sick for helping us to extract this ephemeris data.

that is either 6 hours before or 6 hours after the time given by ϕ. Going through this, we see that the peak price hour for the day is 15.2, or about 3 pm.

Similarly, we can interpret the annual seasonal. Note that the amplitude of the swings is less than for the daily swings (a maximum factor of 1.16 at the peak). The peak month is month 10.86, which is late October to early November.[5] The annual seasonal in electricity is clearly affected by several confounding factors: holiday seasons, air conditioning seasons, and heating seasons. We have already removed the effect of darkness hours for street lighting.

5.3 Estimating Mean Reversion of Prices

We take the residuals of the previous regression as estimates of the deseasonalized and detrended variable X_t. We represent the mean-reverting diffusion (5) as an auto-regression of changes in X_t on prior levels X_{t-1}:

$$X_t - X_{t-1} = \kappa\mu - \kappa X_{t-1} + \sigma\Delta\omega_t \tag{24}$$

A priori there is no reason to believe that the coefficient of mean reversion, κ will differ according to the number of generators on-line, but we would expect the long-run mean of the log price, $\boldsymbol{\mu}$ to vary with the number of large generators that are on line. Thus, we record this auto-regression in Table 5 for the generator states Gen 9, Gen 10, ..., Gen 14. The autoregression coefficient κ is always significantly different from zero, so the process is not a random walk (it doesn't have a unit root). We report the half life of a deviation from the mean, measured in hours. Note that half of any price shocks are eroded away withing 4 or 5 hours. This reduces the price risk considerably. We also report the long-run mean $\boldsymbol{\mu}$ and the exponential of the long run mean. This latter number provides a price factor adjustment, according to the number of large generators that are running. Thus, when only 9 large generators are on-line, the price is about 68% higher than nor-

[5]Of the 4 years in our data set, two Octobers were very unusual. October 1998 had a shortage of supply and some cities had to shed load by cutting off electricity to customers. October 1999 saw a change in the rules for revising bids to make it harder for a generator to bid power in but withdraw it and re-bid at a higher price if demand is high. The rule transition caused some difficulties in the market. Both Octobers saw high electricity prices and this even might not repeat itself.

Table 5: Estimation of the Mean Reverting Process for Deseasonalized and Detrended Price.

Gen 9	Coef.	Std. Err.	t	Prob $> \|t\|$	Half Life	μ	$\exp(\mu)$
κ	0.200	0.049	4.1	0.00	3.47	0.52	1.68
$\kappa\mu$	0.104	0.047	2.2	0.03			
Gen 10							
κ	0.166	0.021	7.9	0.00	4.18	0.37	1.45
$\kappa\mu$	0.062	0.016	3.8	0.00			
Gen11							
κ	0.191	0.009	21.9	0.00	3.63	0.13	1.13
$\kappa\mu$	0.024	0.005	5.0	0.00			
Gen 12							
κ	0.154	0.005	28.6	0.00	4.49	0.03	1.03
$\kappa\mu$	0.005	0.003	1.8	0.08			
Gen 13							
κ	0.136	0.005	29.6	0.00	5.08	-0.07	0.93
$\kappa\mu$	0.010	0.002	4.5	0.00			
Gen 14							
κ	0.142	0.006	22.5	0.00	4.88	-0.09	0.92
$\kappa\mu$	0.012	0.003	4.0	0.00			

mal, while when 14 generators are running, the price is about 8% lower than normal.

We analyze the residuals of the mean-reversion regressions in Table 6. This provides an estimate of volatility σ, which seems to be fairly constant across the models at 25% to 35% per hour. Thus, price changes are quite volatile on an hourly basis, but deviations decay rapidly over a matter of hours towards the long run mean. The long run mean price is higher when there are fewer generators running.

Table 6 shows that the residuals are quite non-normal, so that the Gauss-Wiener diffusion assumption is violated. The residuals are normally right skewed (except when only 9 generators are running), so it is possible to get large upward deviations in price but not so likely to get large negative deviations in price. Moreover, the distribution is lepto-kurtotic or fat-tailed. This suggests that caution should be exercised in using analytic representations of the state-price density in calculating option prices, since these analytic expressions will generally be based on the assumption of log-normality.

Table 6: Residual Analysis to Estimate Volatility

Num Gen	σ	$\exp(\sigma)$	Kurtosis	Skew	Min	Max	Count
9	0.25	1.29	6.30	-1.90	-1.04	0.43	59
10	0.38	1.46	12.65	0.36	-2.15	2.23	769
11	0.32	1.37	13.91	0.24	-2.06	2.62	4544
12	0.26	1.30	14.92	0.35	-2.28	2.60	9919
13	0.23	1.26	10.16	-0.01	-1.73	1.61	12046
14	0.25	1.28	8.08	0.32	-1.45	2.53	6773

An alternative approach to pricing options that may be preferred is to simulate the Markov process for Z_t using the Markov transition matrix, and then simulate the conditional changes in the deseasonalized and detrended price X_t conditional on the values of Z_t. Then the detrended price can be multiplied by the seasonal factors to get a final price. Repeating the simulation for 1000 or 10,000 times will give a good representation of the state price density, which can be used to form the expectations of equations (15) and (18) to price out European calls and other options. This is the subject of future work.

Figure 2 shows such simulation looking one month ahead to the deseasonalized price for January 30, 2000. The maximum price that the system computers can handle is $999. Rarely does the price go above $200, but it often exceeds $50.

Figure 3 zooms in to the pricing region of $0 to $200.

6 Concluding Remarks

In this paper we have analyzed the generation and pricing process for the Alberta Power Pool, which is an electricity market with minimal excess capacity and minimal opportunities for importation of electricity. The Pool has a number of large generators, which develop most of the power, so that prices can spike upward when these large generators go off-line.

Over the 4-year period we studied, there were always 9 to 14 of these generators running. The long run mean price is higher when fewer generators are running, by a factor of approximately 1.7 when only 9 generators are running. There are significant price variations within the day, with prices increasing by a factor of 1.6 for the daily peak. Price deviations are mean

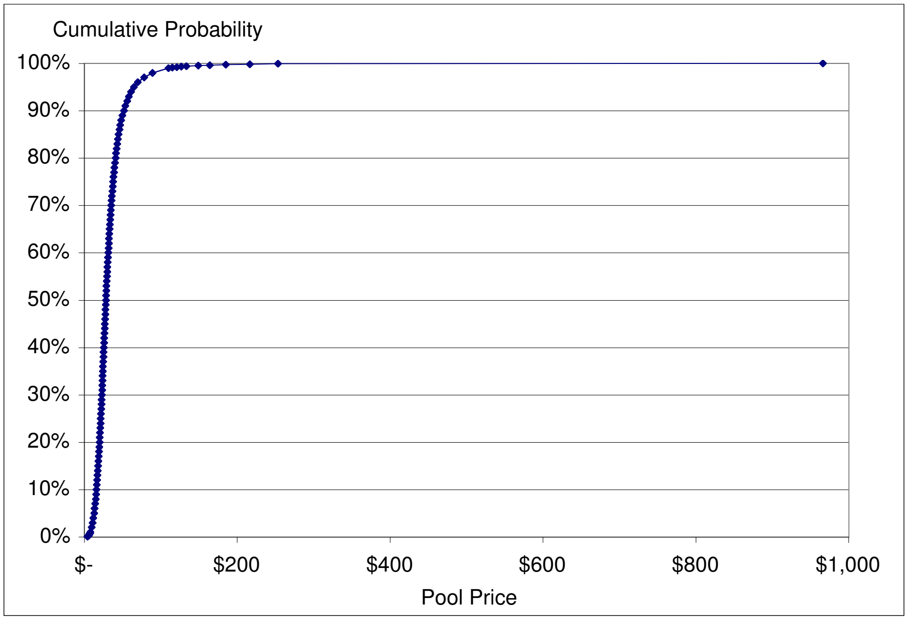

Figure 2: Bootstrap simulation of January 30 Pool Price distribution, starting at the means on January 1.

reverting with a half life of deviations from the long-run mean being measured in hours rather than days. However, the hourly volatility of prices is high—on the order of 25% to 35%. Another important factor in the pricing of electricity is that deseasonalized prices almost tripled over the 4-year period. It is unlikely that electricity prices will continue to grow at this rate, so some careful judgement is required to use the model on a forward-looking basis to price electricity derivatives.

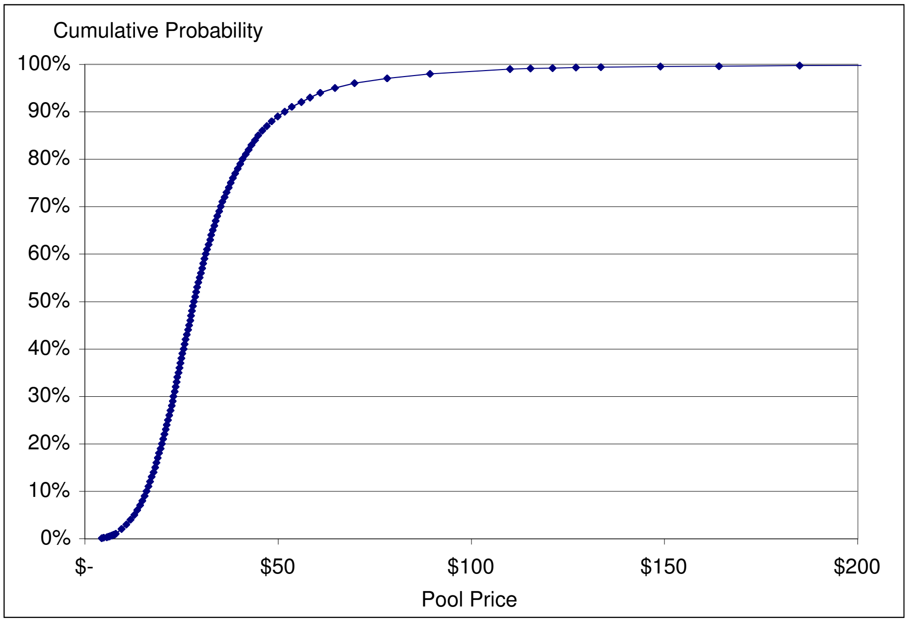

Figure 3: Closeup view of restricted price range in simulation of January 30 Pool Price distribution, starting at the means on January 1.

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